Integrability out of equilibrium: current, fluctuations, correlations

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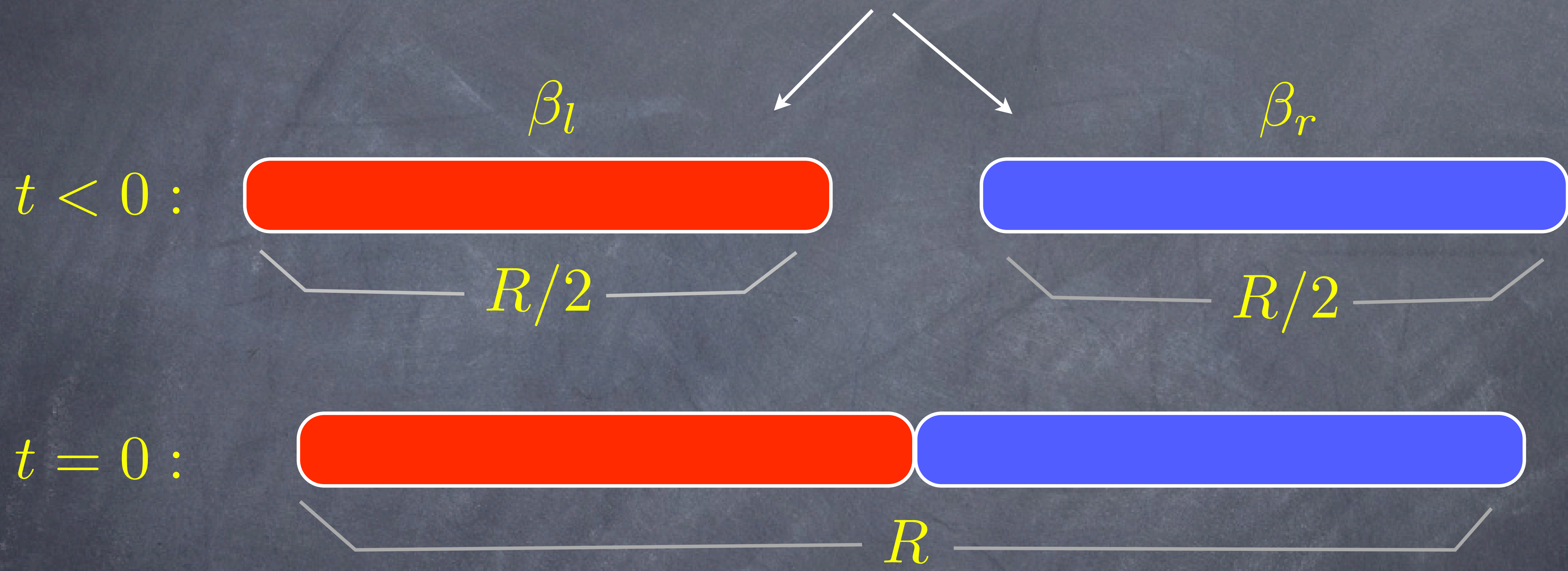
My works on this subject

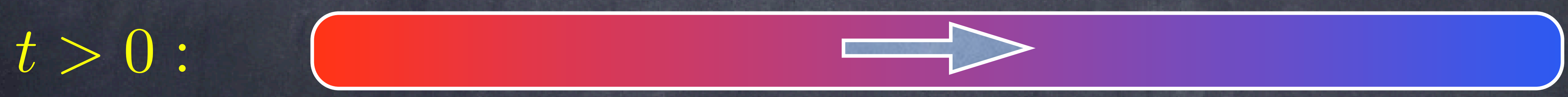
J. Phys. A 45 (2012) 362001 with Denis Bernard
J. Phys. A 46 (2013) 372001 with Denis Bernard
arXiv:1212.1077, to appear in Les Houches Lecture Notes
arXiv:1302.3125 with Denis Bernard, accepted in Ann. Inst. H. Poincaré
arXiv:1305.0518 with Yixiong Chen
arXiv:1305.4984 with A. De Luca, J. Viti and D. Bernard
arXiv:1306.3192 with Marianne Hoogeveen and Denis Bernard
work in progress with O. A. Castro Alvaredo, Y. Chen, M. Hoogeveen

Physical situation

Independent quantum chains
in different thermal states

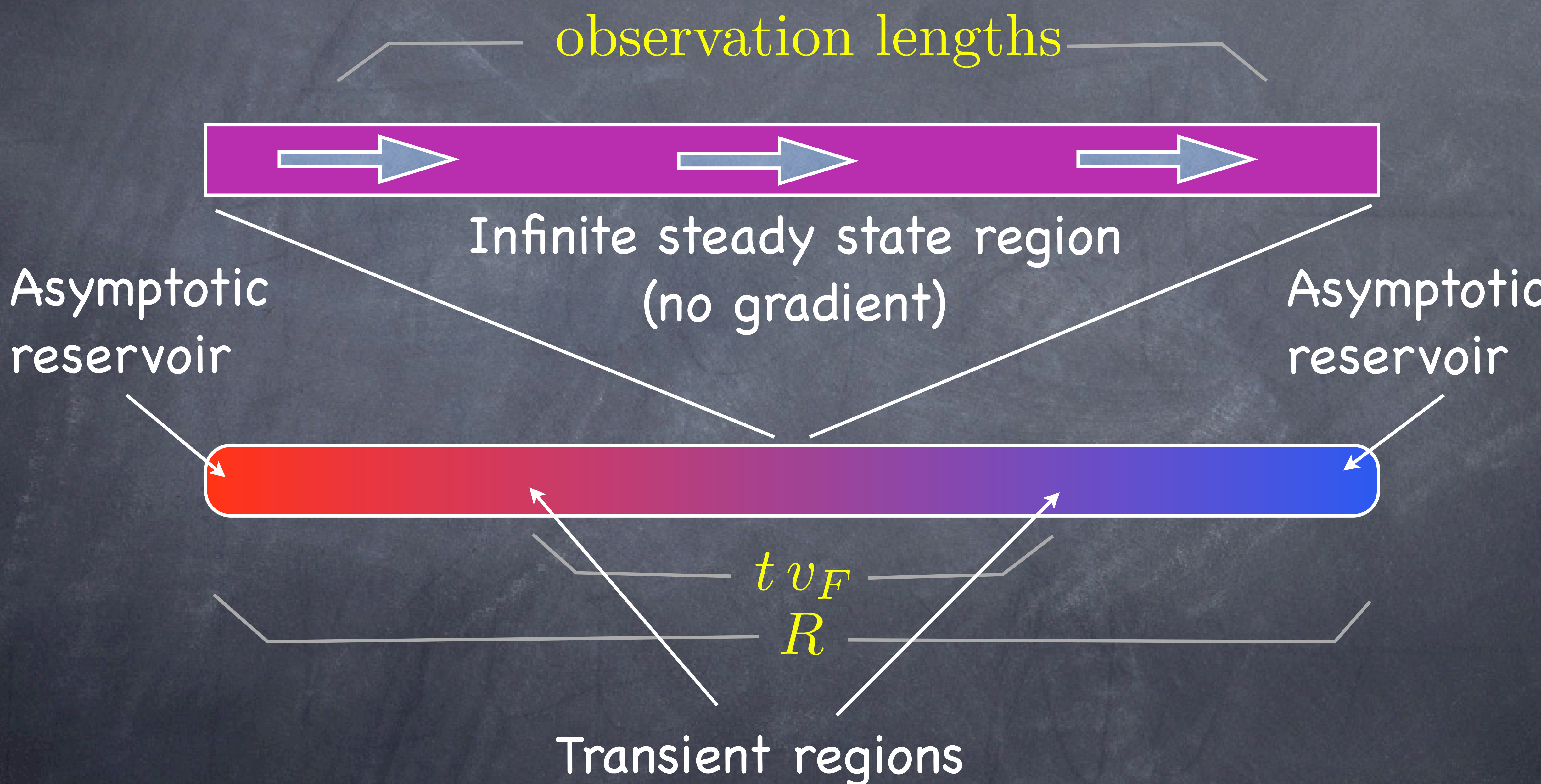


Unitary evolution....



Steady-state limit

$$R \gg t\, v_F \gg \text{observation lengths}$$

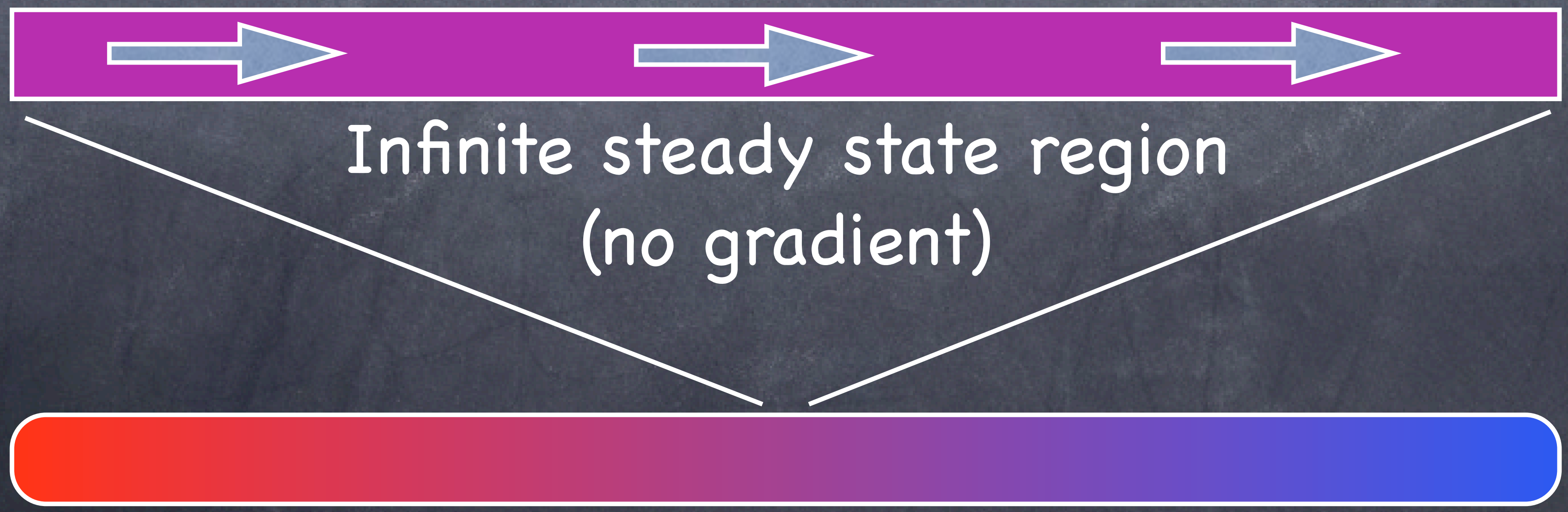


Anything happens?

Generic non-integrable models: Fourier's law says that no gradient => no current; thermal equilibrium in steady state region. **Nothing.**

Integrable models: conservation laws allow a flow of energy to exist; non-equilibrium steady state. **Something interesting!**

Any critical point: collective behaviours allow a flow of energy; non-equilibrium steady state. **Something interesting!** [D. Bernard & BD, 2012]



How / what to calculate

Initially: $\rho_0 = e^{-\beta_l H^l - \beta_r H^r}$

Evolution Hamiltonian: $H = H^l + H^r + H_{\text{contact}}$

Steady state limit:

$$\langle \cdots \rangle_{\text{ness}} = \lim_{t_0 \to -\infty} \lim_{R \to \infty} \frac{\mathrm{Tr}\left(e^{iHt_0} \rho_0 e^{-iHt_0} \ldots\right)}{\mathrm{Tr}\left(\rho_0\right)}$$

↓

Observables supported on a finite region

Current: $J(\beta_l, \beta_r) = \langle \mathcal{J} \rangle_{\text{ness}}, \quad \mathcal{J} = \frac{d}{dt} \frac{H^r - H^l}{2} = i[H, \frac{H^r - H^l}{2}]$

Correlations: $\langle \mathcal{O}(x_1, t_1) \mathcal{O}(x_2, t_2) \rangle_{\text{ness}}$

Fluctuations of energy transfer:

$$Q = \frac{1}{2}\left(H^r - H^l\right)$$

$$Q = q_0$$

$$Q = q_0 + q$$

$$\Omega_t(q) = \text{probability of } q \text{ at time } t$$

$$= \sum_{q_0} \mathrm{Tr}\left(P_{q_0+q} e^{-iHt} P_{q_0}\ \rho_0\ P_{q_0} e^{iHt} P_{q_0+q}\right)$$

$$F(z) = \text{generating function of scaled cumulants}$$

$$= \sum_n \frac{z^n}{n!} \lim_{t\to\infty} \frac{1}{t} \langle q^n \rangle_{\Omega_t}^{\text{cumulant}} = \sum_n \frac{z^n}{n!} C_n$$

Fluctuations of energy transfer:

It turns out that:

$$F(z) = \lim_{t\to\infty} \frac{1}{t} \log\left[\langle e^{zQ(t)} e^{-zQ}\rangle_{\rm ness}\right], \quad Q(t) = e^{iHt} Q e^{-iHt}$$

charge transfer in free fermion models; indirect measurement protocol:
[Levitov & Lesovik 1993; Klich 2002; Schonhamer 2007; D. Bernard & BD 2012]
energy transfer in CFT:
[D. Bernard & BD 2012]

Surprisingly, for energy transfer in integrable models:

$$\boxed{F(z) = \int_0^z dz'\, J(\beta_l - z', \beta_r + z')}$$

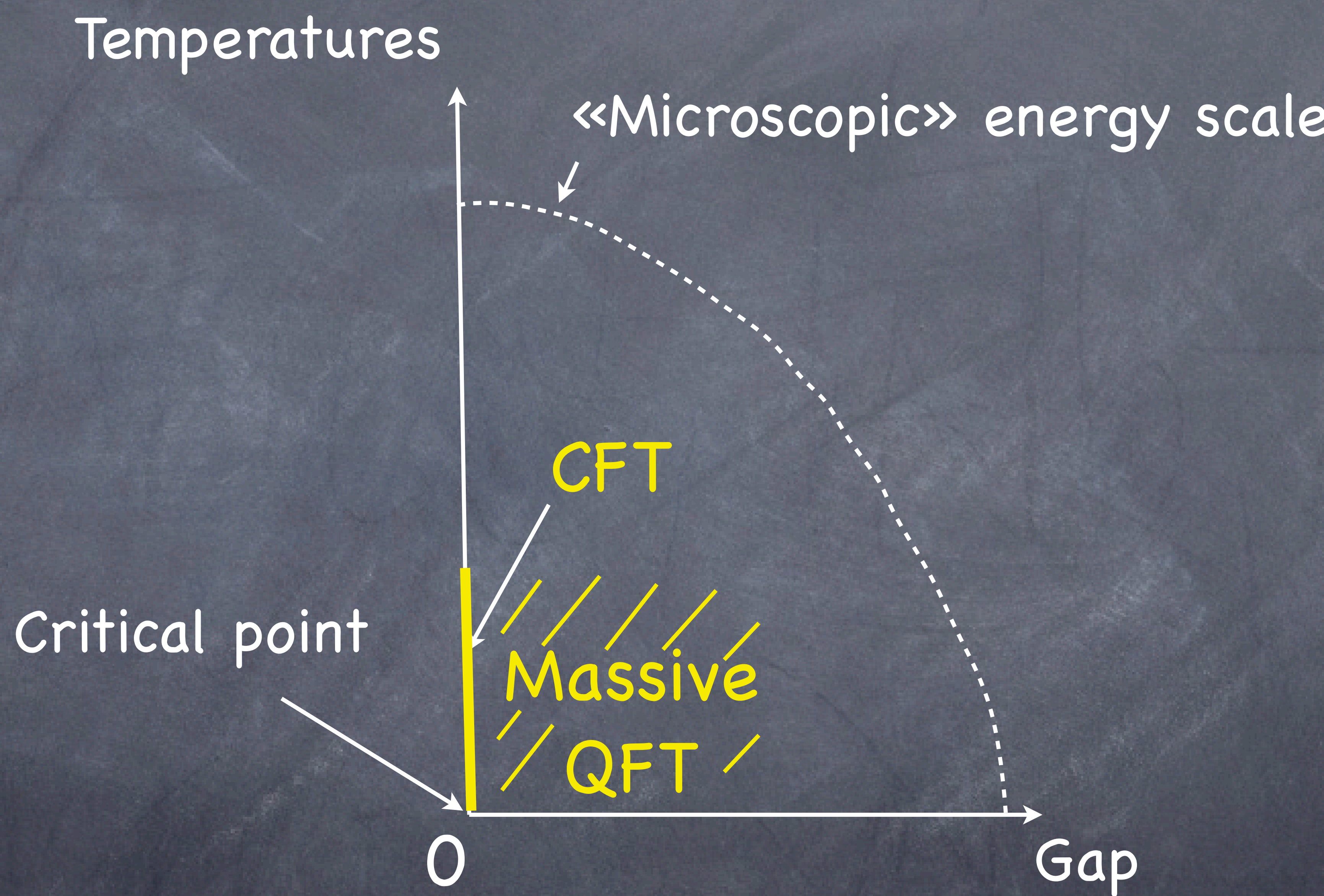
[D. Bernard & BD 2013]

holds whenever there is «pure transmission» SQ = -QS
where S is the scattering matrix. [D. Bernard & BD 2013]

implies the standard fluctuation relations of Jarzynski, Wojcik based on Gallavotti; see Esposito, Harbola, Mukamel (RMP 2009).

$$F(z) = F(\beta_l - \beta_r - z)$$

Scaling limit: (relativistic) QFT



Formal description of steady state

in massive QFT

[D. Bernard & BD 2012]
[BD 2012]

$$\langle \cdots \rangle_{\rm ness} = {\rm Tr}\left(e^{-W}\cdots\right)\ /\ {\rm Tr}\left(e^{-W}\right)$$

$$W = \beta_l \int_0^\infty d\theta E_\theta n_\theta + \beta_r \int_{-\infty}^0 d\theta E_\theta n_\theta$$

Total energy of right-moving asymptotic particles

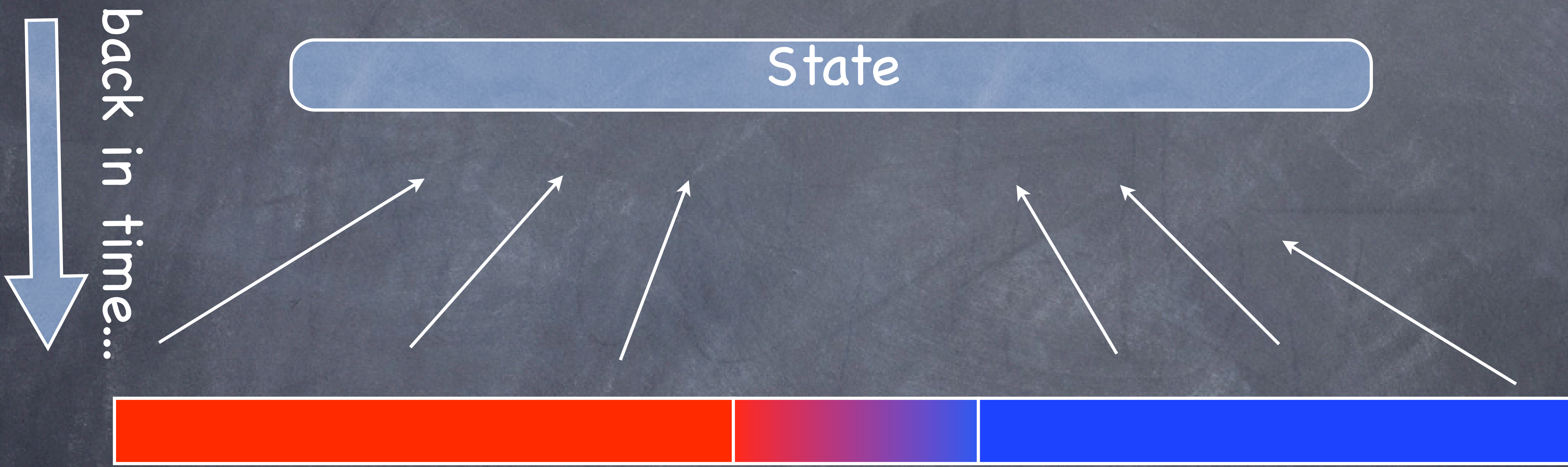
Total energy of left-moving asymptotic particles

*In massive QFT: states are described by asymptotically free particles characterized by their rapidities

Formal description of steady state

in massive QFT

[D. Bernard & BD 2012]
[BD 2012]



agreement with free fermion calculations (XY model)
[W. H. Aschbacher & C.-A. Pillet, 2003]

Formal description of steady state

in CFT

[D. Bernard & BD 2012, 2013]

$$\langle \cdots \rangle_{\text{ness}} = \mathrm{Tr}\left(e^{-W} \cdots\right) \ / \ \mathrm{Tr}\left(e^{-W}\right)$$

$$W = \beta_l \frac{L_0}{2\pi R} + \beta_r \frac{\bar{L}_0}{2\pi R} \qquad (R \to \infty)$$

Total energy of right-movers — Total energy of left-movers

*In CFT: densities separate into right-movers and left-movers

energy density: $h(x) = h_+(x) + h_-(x)$

momentum density: $p(x) = h_+(x) - h_-(x)$

Current / fluctuations in CFT

[D. Bernard & BD 2012, 2013]

$$J(\beta_l, \beta_r) = \frac{\pi c}{12}(\beta_l^{-2} - \beta_r^{-2}) = \frac{\pi c k_B^2}{12\hbar}(T_l^2 - T_r^2)$$

central charge of Virasoro algebra

$$h_+(x) \propto -\frac{c}{24} + \sum_{n \in Z} L_n e^{-\frac{2\pi i n x}{R}}$$

Virasoro

- confirmed by numerics

[C Karrasch, R. Ilan & J. E. Moore, 2012]

- agreement with Luttinger liquid results

[M. Mintchev & P. Sorba, 2012]

[D. B. Gutman, Yu. Gefen & A. D. Mirlin, 2010]

Current / fluctuations in CFT

[D. Bernard & BD 2012, 2013]
[BD, M. Hoogeveen & D. Bernard 2013]

$$F(z) = f(z;\beta_l) + f(-z;\beta_r) \qquad f(z;\beta) = \frac{c\pi}{12}\left(\frac{1}{\beta - z} - \frac{1}{\beta}\right)$$

Recall: $F(z) = \text{generating function of scaled cumulants}$

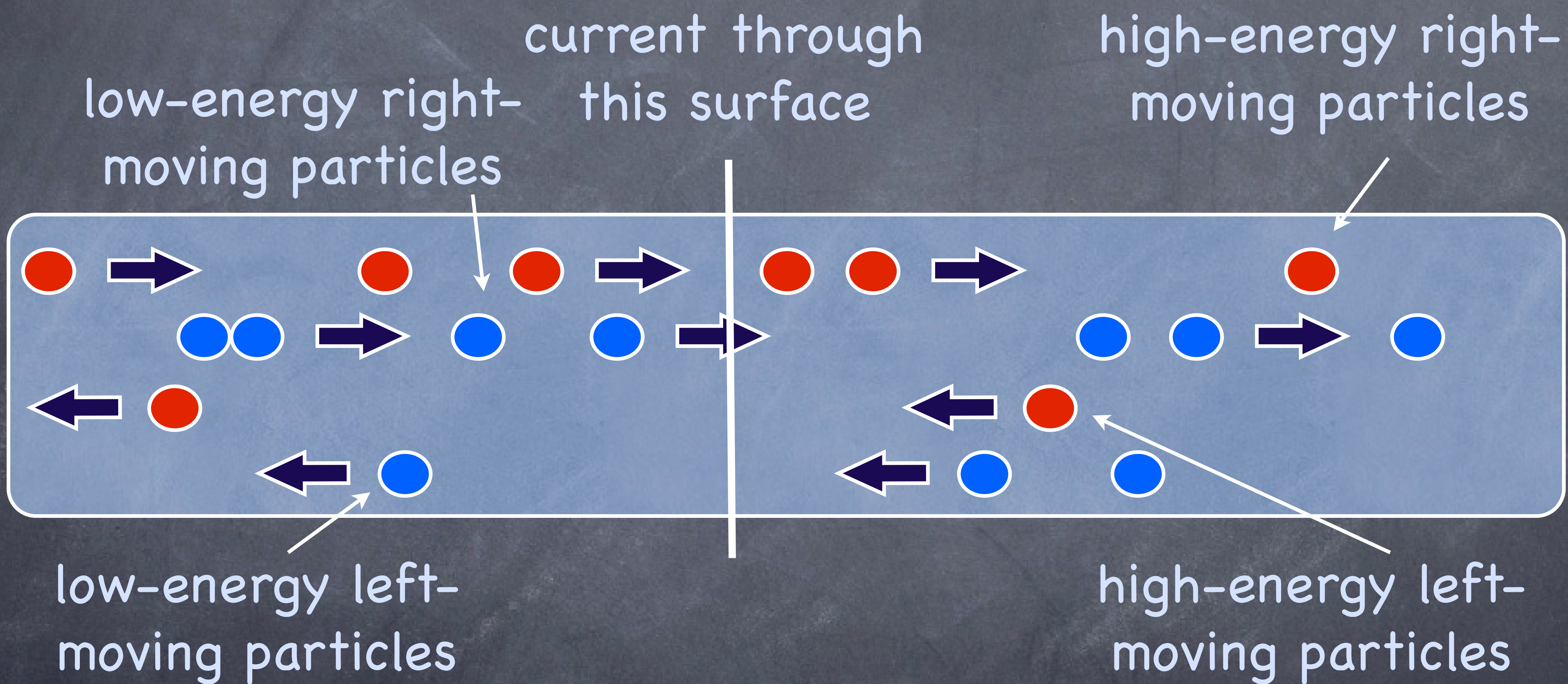
Independent Poisson process for every energy value: independent energy packets, travelling in both directions

$$F(z) = \int dq\, \omega(q)\, (e^{zq} - 1), \qquad \omega(q) = \frac{c\pi}{12} \cdot \left\{ \begin{array}{ll} e^{-\beta_l q} & (q > 0) \\ e^{\beta_r q} & (q < 0) \end{array} \right)$$

$\omega(q)$ → intensity

$(e^{zq} - 1)$ → cumulant generating function for a single Poisson process

Poisson process interpretation



Current / fluctuations Ising model

$$W = \int d\theta\, w(\theta)\, a^\dagger(\theta) a(\theta), \quad w(\theta) = m\cosh\theta \cdot \begin{cases} \beta_l & (\theta > 0) \\ \beta_r & (\theta < 0) \end{cases}$$

The current $\mathcal{J}$ is quadratic in the creation and annihilation operators, so a rather simple free-fermion calculation gives

$$J = \frac{\mathrm{Tr}\left(e^{-W}\mathcal{J}\right)}{\mathrm{Tr}\left(e^{-W}\right)} = \frac{m^2}{4\pi}\int d\theta\, \frac{\sinh 2\theta}{1 + e^{w(\theta)}}$$

=> **Landauer form**: p dp times thermal distributions on left / right

Fluctuations: **Independent Poisson processes**,

$$\omega(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2\pi n^2} \cdot \begin{cases} e^{-\beta_l q} & (q > nm) \\ e^{\beta_r q} & (q < -nm) \end{cases}$$

using $F(z) = \int_0^z dz'\, J(\beta_l - z', \beta_r + z') \ = \int dq\, \omega(q)\, (e^{zq} - 1)$

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

We extend the thermodybamic Bethe ansatz [Al. B. Zamolodchikov, 1990] to the non-equilibrium steady state density matrix:

$$J = \frac{\mathrm{Tr}\left(e^{-W} p(0)\right)}{\mathrm{Tr}\left(e^{-W}\right)}$$

current operator = momentum density

In order to calculate the trace, we make the system finite and periodic, replacing the operator W with an approximate finite-length version W^L. Following Zamolodchikov, a state $|v\rangle$ is described by Bethe roots with energies and momenta that approximate the relativistic ones. So we set:

$$W^L|v\rangle = \sum_k w_k|v\rangle, \quad w_k = e_k \cdot \begin{cases} \beta_l & (p_k > 0) \\ \beta_r & (p_k < 0) \end{cases}$$

$$|v\rangle : \quad \text{energies } e_k, \quad \text{momenta } p_k$$

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

Using translation invariance and taking the large length limit, we get:

$$J = \lim_{L\to\infty} L^{-1} \frac{\mathrm{Tr}_L\left[e^{-W^L} P\right]}{\mathrm{Tr}_L\left[e^{-W^L}\right]}$$

P → total momentum operator

We define a «free energy» and calculate the current from it:

$$f^a := -\lim_{L\to\infty} L^{-1} \log \mathrm{Tr}_L\left[e^{-W^L - aP}\right], \quad J = \frac{d}{da} f^a \Big|_{a=0}.$$

We can then use the methods of the thermodynamic Bethe ansatz…

[Al. B. Zamolodchikov, 1990]

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

$$J = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{m \cosh\theta \; x(\theta)}{1 + e^{\epsilon(\theta)}}$$

$$x(\theta) = m \sinh\theta + \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} \frac{\varphi(\theta - \gamma)\, x(\gamma)}{1 + e^{\epsilon(\gamma)}}$$

dressed momentum

$$\epsilon(\theta) = w(\theta) - \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} \varphi(\theta - \gamma) \log(1 + e^{-\epsilon(\gamma)})$$

non-equil. pseudo energy

where we recall that

$$w(\theta) = m \cosh\theta \cdot \begin{cases} \beta_l & (\theta > 0) \\ \beta_r & (\theta < 0) \end{cases}$$

driving term with a jump

$$\varphi(\theta) = -i \frac{d}{d\theta} S(\theta)$$

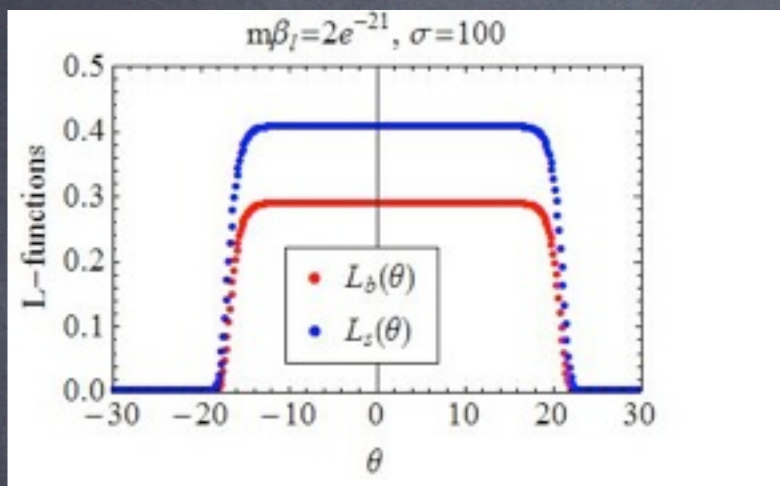
differential scattering phase

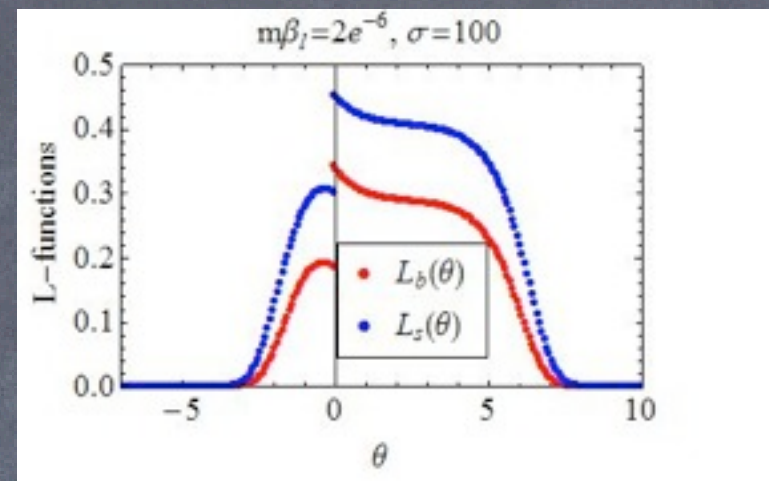
Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

L-function has a jump and an asymmetry

$$L(\theta) = \log(1 + e^{-\epsilon(\theta)})$$





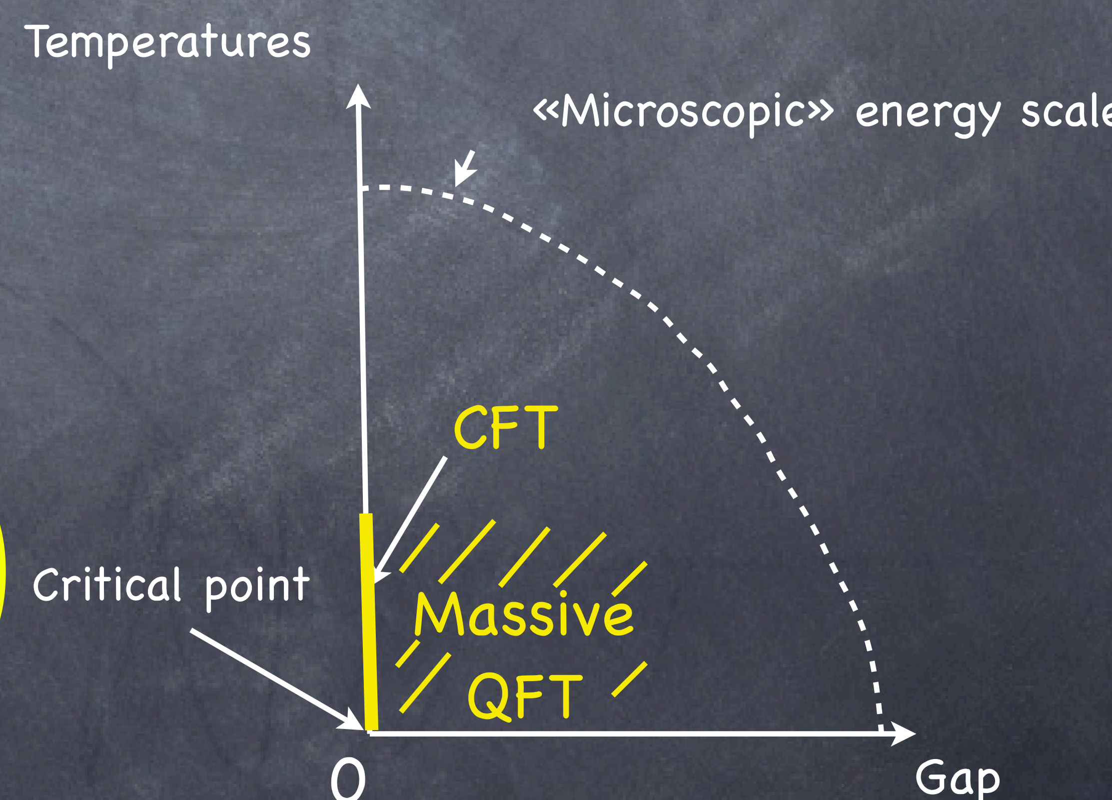
Sine-Gordon soliton and breather L-functions at high and medium temperatures

Large temperature / small gap limit: we recover the exact CFT result

$$J(\beta_l, \beta_r) \sim \frac{\pi c}{12}(T_l^2 - T_r^2)$$

$$c = \frac{3}{\pi^2}\int_{\epsilon(0)}^{\infty} d\epsilon \left(\log(1 + e^{-\epsilon}) + \frac{\epsilon}{1 + e^{\epsilon}} \right)$$

An expression for the central charge [Al. B. Zamolodchikov, 1990]

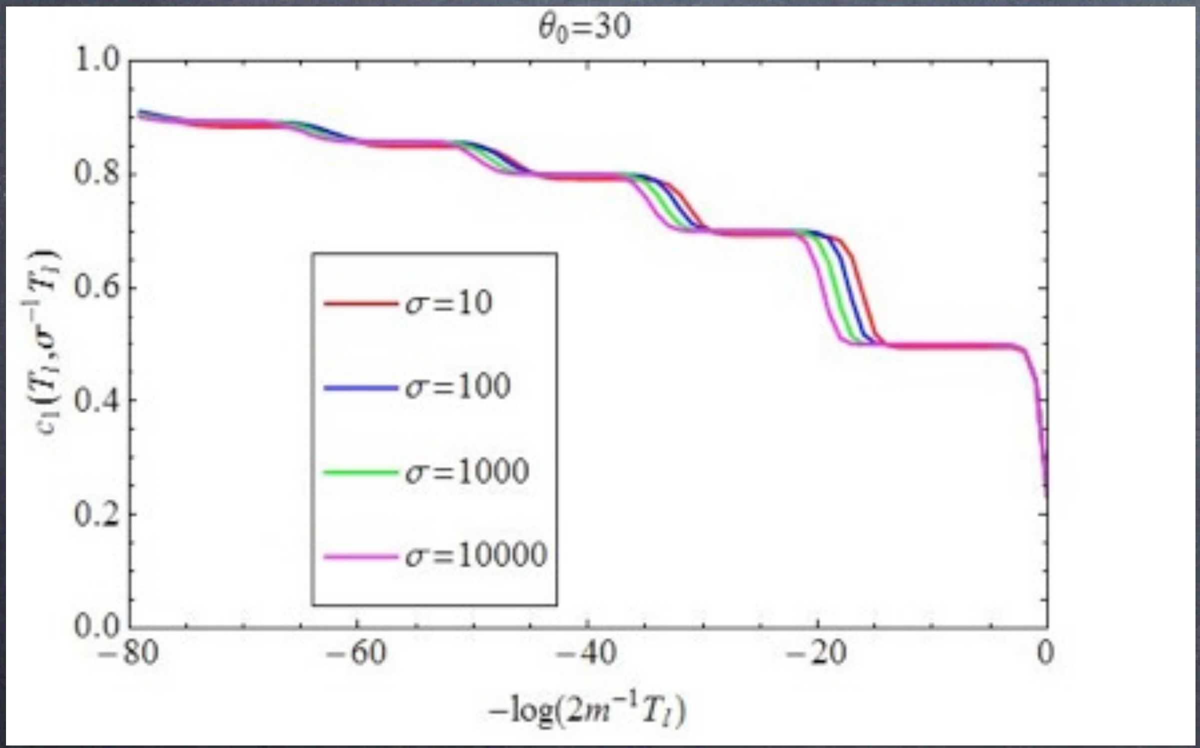


Non-equilibrium c-functions

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

Dividing current by CFT form, get something that equates the central charge at fixed points. In fact, it is monotonic between fixed points: a **c-function** [A. B. Zamolodchikov 1986, Al. B. Zamolodchikov 1990], **which measures the number of degrees of freedom along an RG trajectory.**

$$c_1(T_l, T_r) = \frac{J(\beta_l, \beta_r)}{(\pi/12)(T_l^2 - T_r^2)}$$



Non-equilibrium c-function in Al. B. Zamolodchikov's roaming trajectories model

Non-equilibrium c-functions

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

It turns out that **all cumulants give rise to c-functions**:

$$c_n(T_l, T_r) = \frac{12 C_n(T_l, T_r)}{\pi n! \, (T_l^{n+1} + (-1)^n T_r^{n+1})}$$

Explanation: by lowering the mass, keeping the temperatures the same, we allow more excitations to be present, hence there is more current, and more fluctuations

$$\frac{d}{dm} C_n \leq 0.$$

Poisson process interpretation

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

We can calculate the Poisson process intensity by Fourier transform of the current; it is an intensity **if and only if it is positive:**

$$\omega(q) = \frac{1}{q}\int \frac{d\lambda}{2\pi}\, J(\beta_l - i\lambda, \beta_r + i\lambda)\, e^{-i\lambda q}.$$

using $F(z) = \int_0^z dz'\, J(\beta_l - z', \beta_r + z') \;=\; \int dq\, \omega(q)\, (e^{zq} - 1)$

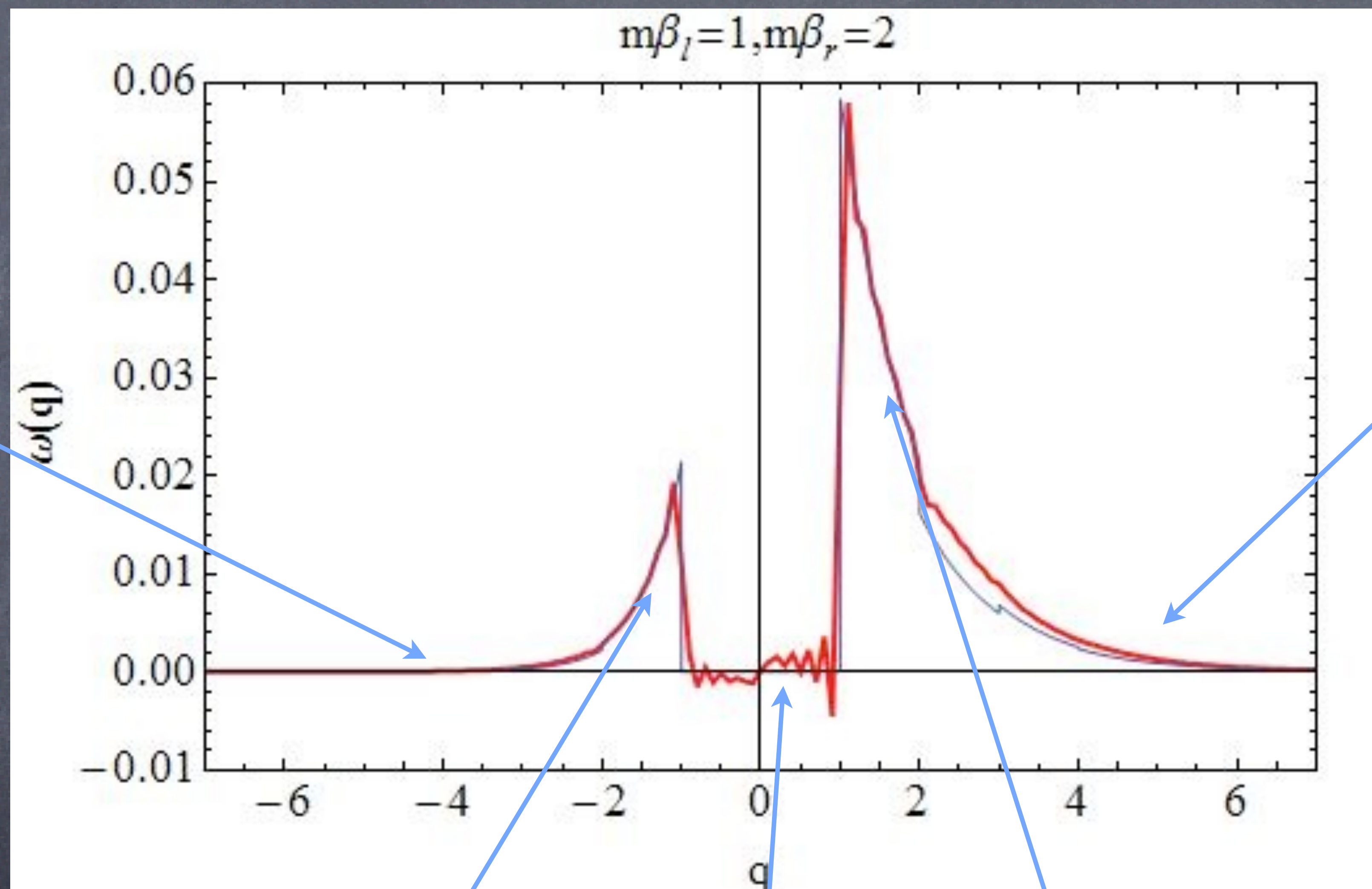
$$\text{Poisson processes iff} \quad \omega(q) \geq 0$$

Poisson process interpretation

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

It turns out that indeed, this Fourier transform is positive!

Asymptotically
CFT c=1
$\frac{\pi}{12}e^{\beta_r q}$



Asymptotically
CFT c=1
$\frac{\pi}{12}e^{-\beta_l q}$

Exactly Ising
(one-particle)

Exactly 0

Exactly Ising
(one-particle)

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]
extending: [BD 2005]

We use a «spectral decomposition» in the space of operators («Liouville space») in order to obtain a convergent expansion valid at large distances, following an old idea in C* algebra.

[Gelfand & Naimark 1943, Segal 1947]

Inner product on the space of operators with respect to a density matrix:

$$ {}^{\rho}\langle A|B\rangle^{\rho} = \frac{\mathrm{Tr}\left(\rho\, A^{\dagger} B\right)}{\mathrm{Tr}\left(\rho\right)} \qquad A, B \in \mathrm{End}(\mathcal{H}) $$

Choice of basis states:

creation / annihilation operators

$$ |\mathrm{vac}\rangle^{\rho} := |\mathbf{1}\rangle^{\rho} $$

$$ |\theta_1, \ldots, \theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N} := Q^{\rho}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)\, |a^{\epsilon_1}(\theta_1)\cdots a^{\epsilon_N}(\theta_N)\rangle^{\rho} $$

convenient normalization: $Q^{\rho}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) := \prod_{i=1}^{N}\left(1 + e^{-\epsilon_i W(\theta_i)}\right).$

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

The spectral decomposition is:

$$\frac{\mathrm{Tr}\left[\rho\,\sigma(x)\sigma(0)\right]}{\mathrm{Tr}[\rho]} = {}^{\rho}\langle\mathrm{vac}|\sigma(x)\sigma(0)\rangle^{\rho}$$

left-action: $A^{\ell}|B\rangle^{\rho} = |AB\rangle^{\rho}$

$$= {}^{\rho}\langle\mathrm{vac}|\sigma^{\ell}(x)\sigma^{\ell}(0)|\mathrm{vac}\rangle^{\rho}$$

$$= \sum_{\mathrm{states}} {}^{\rho}\langle\mathrm{vac}|\sigma^{\ell}(x)|\mathrm{state}\rangle^{\rho}\;{}^{\rho}\langle\mathrm{state}|\sigma^{\ell}(0)|\mathrm{vac}\rangle^{\rho}$$

Form factor expansion!

We use $\rho = e^{-W}$ with $W = \int d\theta\, w(\theta)\, a^{\dagger}(\theta)a(\theta)$, and we take $\sigma(x)$ to be the Ising spin field (say in ordered phase).

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

Mixed-state form factors are defined as:

$$f^{\rho}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) := {}^{\rho}\langle \mathrm{vac}|\sigma^{\ell}(0)|\theta_1,\ldots,\theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N}.$$

Results:

direction of branch of twist field (right or left)

expectation value

$$f^{\rho;\pm}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) = h^{\pm}_{\epsilon_1}(\theta_1) h^{\pm}_{\epsilon_2}(\theta_2) \left(\tanh\frac{\theta_2-\theta_1 \pm i(\epsilon_2-\epsilon_1)\mathbf{0}}{2}\right)^{\epsilon_1\epsilon_2} \langle\sigma_{\pm}\rangle_{\rho}$$

«leg factors»

vacuum form factor

$$h^{+}_{\epsilon}(\theta) = \sqrt{\frac{i}{2\pi}}\, e^{\epsilon\frac{i\pi}{4}}\, g(\theta+\epsilon i\mathbf{0})^{\epsilon}, \quad h^{-}_{\epsilon}(\theta) = -ih^{+}_{-\epsilon}(\theta)$$

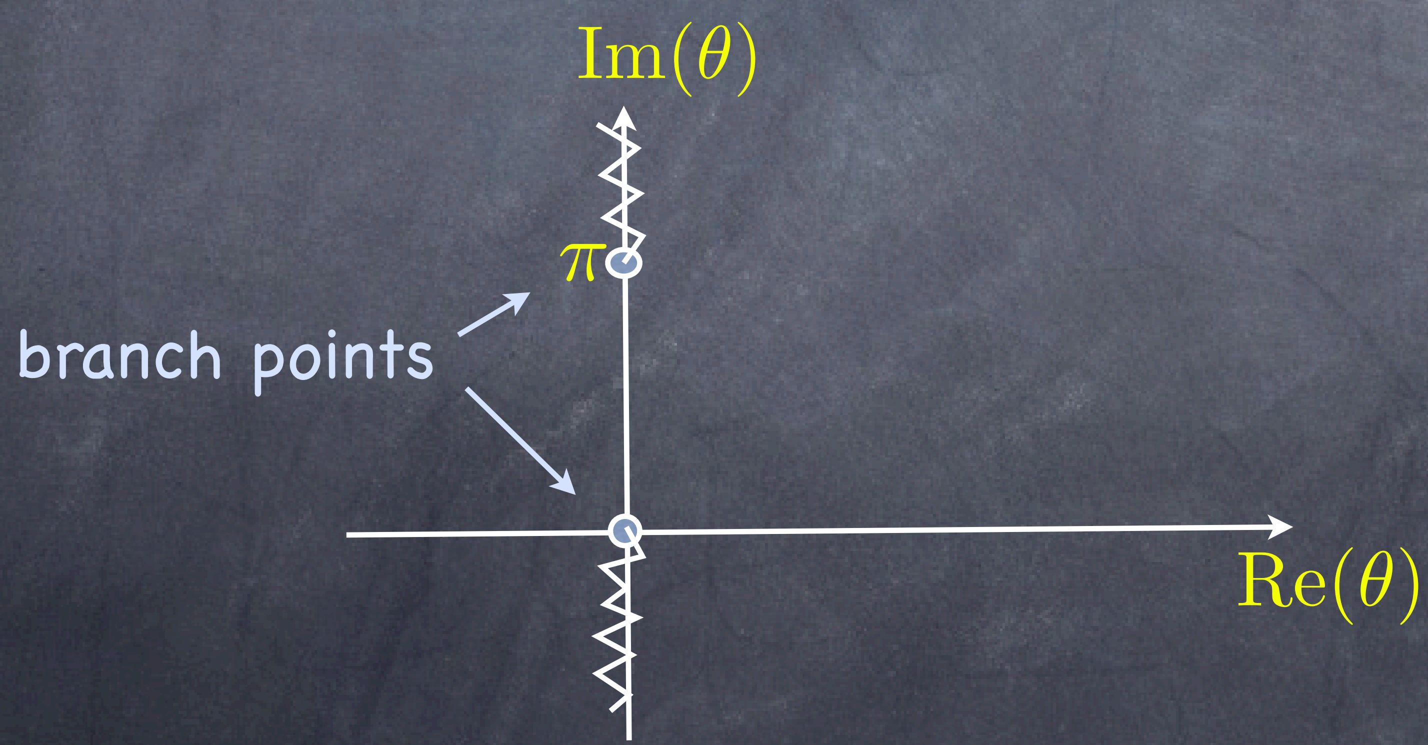
$$g(\theta) = \exp\left[\int \frac{d\theta'}{2\pi i}\log\left(\tanh\frac{\theta-\theta'+i\mathbf{0}}{2}\right)\frac{d}{d\theta'}\log\left(\tanh\frac{w(\theta')}{2}\right)\right]$$

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

Analytic structure of leg factors determines leading asymptotic behaviour of two-point function. This reproduces the correct asymptotic in the GGE occurring after a magnetic-field quench as calculated by Calabrese, Essler and Faggoti (2011-2012).

In the non-equilibrium steady state: there are branch points:



Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

We find an oscillatory behaviour at large distances, e.g. in the disordered phase:

$$\frac{e^{-x\mathcal{E}_{\text{ness}}}}{mx}\cos(\nu\log(mx)+B)$$

with a frequency determined by the temperatures:

$$\nu=\frac{1}{\pi}\log\left(\coth\frac{m}{2T_r}\;\tanh\frac{m}{2T_l}\right)$$

Conclusion and perspectives

- Generalization to presence of impurities
- Current and fluctuations in integrable spin chains
- Non-equilibrium correlation functions / mixed-state form factors in integrable models
- Higher dimensions (in progress - AdS/CFT...)