Correlation functions of the cyclic SOS model from algebraic Bethe Ansatz
Thermodynamic limit

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Articles:
Computation of correlation functions within ABA

For $|\psi_g\rangle$ a ground state of the cyclic SOS model for which $\eta = \frac{r}{L}$,

$$\langle \psi_g | O | \psi_g \rangle$$

$O = \delta_s, \sigma^z_m, \delta_s E^{\alpha_1 \alpha_1}_1 \ldots E^{\alpha_m \alpha_m}_m, \ldots$

Diagonalization achieved by ABA Felder, Varchenko (1996):

$|\{u\}, \omega\rangle: s \mapsto \varphi(s)B(u_1; s)B(u_2; s-1) \ldots B(u_n; s-n+1)|0\rangle \in \text{Fun}(\mathcal{H}[0])$

for which $\varphi(s) = \omega^s \prod_{j=1}^{n} \frac{[1]}{[s-j]}$ and with $\{u_1, \ldots, u_n\}$, solution of

$$a(u_j) \prod_{l \neq j} \frac{[u_l - u_j + 1]}{[u_l - u_j]} = (-1)^{rk} \omega^{-2} d(u_j) \prod_{l \neq j} \frac{[u_j - u_l + 1]}{[u_j - u_l]}$$

with $N = 2n + kL$ ($k \in \mathbb{Z}$) and $\omega^L = (-1)^m$ (for $\eta = r/L$).

To characterize $|\psi_g\rangle$ this talk

Quantum inverse problem solved similarly as for the XXZ spins chain

Single determinant representation for the scalar product between a Bethe eigenstate and a Bethe arbitrary state (in particular for the norm, and for the form factors)

$\mapsto$ partial scalar product expressed as a sum of $L$ terms because of the dynamical parameter

To take the thermodynamic limit ($N \to \infty$) this talk
Ground states:
It is more convenient to work with conjugate periods \( \tilde{\eta} = -\frac{n}{\tau}, \tilde{\tau} = -\frac{1}{\tau} \).

- Jacobi’s imaginary transformation: \([u] \propto e^{i\pi\eta\tilde{\eta}u^2}\theta_1(x = \tilde{\eta}u; \tilde{\tau})\)
- Logarithmic Bethe ansatz equations for \((\{x_j\}_{1 \leq j \leq n}, \omega_x = \exp(i\pi\frac{rn + 2\ell_x}{L}))\):

\[
Np_0(x_j) - \sum_{l=1}^{n} \vartheta(x_j - x_l) = 2\pi \left( n_j - \frac{n + 1}{2} + \frac{rn + 2\ell_x}{L} + 2\eta \sum_{l=1}^{n} x_l \right), 1 \leq j \leq n
\]

with \(p_0(z) = i \log \frac{\theta_1(\tilde{\eta}/2 + z)}{\theta_1(\tilde{\eta}/2 - z)}\) (bare momentum), \(\vartheta(z) = i \log \frac{\theta_1(\tilde{\eta} + z)}{\theta_1(\tilde{\eta} - z)}\) (bare phase), \(n_j \in \mathbb{Z}\).

- When \(\eta\) is rational \((\eta = r/L)\), in the domain \(0 < \eta < \frac{1}{2}\) and for \(N \to +\infty\), all roots \(x_j\) of the ground states, for which \(n = N/2\) and \(n_{j+1} - n_j = 1\), are real and densely fill \([-1/2; 1/2]\) with density \(\rho(x)\).

- In the thermodynamic limit: \(2(L - r)\) degenerate Bethe ground states, labelled by quantum numbers \(k \in \mathbb{Z}/2\mathbb{Z}\) and \(\ell \in \mathbb{Z}/(L - r)\mathbb{Z}\) (Pearce, Batchelor).

- We now consider two ground states \(|\{x = \tilde{\eta}u\}, \omega_x \rangle = |k_x, \ell_x \rangle\), \(|\{y = \tilde{\eta}v\}, \omega_y \rangle = |k_y, \ell_y \rangle\).

- Sum rules for roots of Bethe ground states?
Finite size corrections

Let $f$ be a 1-periodic and $C^\infty(\mathbb{R})$ function. If $\{x_j\}_{1 \leq j \leq n}$ parametrizes one of the $2(L - r)$ ground states,

$$\frac{1}{N} \sum_{j=1}^{n} f(x_j) = \int_{-1/2}^{1/2} f(z)\rho(z)dz + O(N^{-\infty}).$$

If $g$ is $C^\infty(\mathbb{R})$ function such that $g'$ is 1-periodic,

$$\frac{1}{N} \sum_{j=1}^{n} g(x_j) = \int_{-1/2}^{1/2} g(z)\rho(z)dz + \frac{c_g}{N} \sum_{j=1}^{n} x_j + O(N^{-\infty}),$$

where $c_g = g(1/2) - g(-1/2)$.

This allows us to compute the sum rules while controlling finite size corrections:

- $\{x_j\}, k_x, \ell_x$ and $\{y_k\}, k_y, \ell_y$ two Bethe ground states

$$\sum_{l=1}^{n} (x_l - y_l) = \frac{L(k_x - k_y) + 2(\ell_x - \ell_y)}{2(L - r)} + O(N^{-\infty}),$$

with $k_x, k_y \in \mathbb{Z}/2\mathbb{Z}$, and $\ell_x, \ell_y \in \mathbb{Z}/(L - r)\mathbb{Z}$. 

\[\text{Damien LEVY-BENCHETON} \quad \text{Correlation functions of the SOS model from ABA - CFIM-13}\]
Thermodynamic limit of the form factor

- we want to compute the form factor at the thermodynamic limit between two ground states

\[
\frac{\langle k_x, \ell_x | \sigma^z_m | k_y, \ell_y \rangle}{(\langle k_x, \ell_x | k_x, \ell_x \rangle \langle k_y, \ell_y | k_y, \ell_y \rangle)^{1/2}} = \frac{\langle k_x, \ell_x | \sigma^z_m | k_y, \ell_y \rangle}{\langle k_y, \ell_y | k_y, \ell_y \rangle} \cdot \left( \frac{\langle k_y, \ell_y | k_y, \ell_y \rangle}{\langle k_x, \ell_x | k_x, \ell_x \rangle} \right)^{1/2}
\]

- The norm is expressed as a single determinant representation of size \(n\),

\[
\mathcal{N}^2 \propto \frac{\det_n \left[ \tilde{\Phi} \left( \{y\} \right) \right]}{\det_n \left[ \tilde{\Phi} \left( \{x\} \right) \right]}
\]

- at the thermodynamic limit, the determinant with \(n = N/2\) tends to Fredholm determinants

\[
\det_n \left[ \tilde{\Phi} \left( \{y\} \right) \right] = (-2\pi i \tilde{\eta} N)^n \prod_{l=1}^n \rho(y_l) \left\{ \det \left[ 1 + \hat{K} - \hat{V}_0 \right] + O(N^{-\infty}) \right\}
\]

with integral operators \(\hat{K}\) and \(\hat{V}_0\) acting on \([-1/2, 1/2]\] with respective kernel

\[
K(y - z) = \frac{i}{2\pi} \left\{ \frac{\theta_1'(y-z+\tilde{\eta})}{\theta_1(y-z+\tilde{\eta})} - \frac{\theta_1'(y-z-\tilde{\eta})}{\theta_1(y-z-\tilde{\eta})} \right\}
\]

and \(V_0(y - z) = 2\eta\)

- We eventually find \(\mathcal{N}^2 = \left( \frac{\omega_y}{\omega_x} \right)^{2n} = \left( e^{2i\pi \frac{\ell_y - \ell_x}{L}} \right)^{2n}\)

- \(\langle k_x, \ell_x | \sigma^z_m | k_y, \ell_y \rangle\) expressed as a difference of determinants of size \(n\)
Thermodynamic limit of the form factor

Similarly, $\mathcal{M}$ is expressed as a ratio of Fredholm determinants,

$$
\mathcal{M} \propto \frac{\det\left[1 + (-1)^k \hat{K} + \frac{1-(-1)^k}{2} \hat{V}\right] - \det\left[1 + (-1)^k \hat{K} - \frac{1-(-1)^k}{2} \hat{V}\right]}{\det\left[1 + \hat{K} - \hat{V}_0\right]} + O(N^{-\infty})
$$

with $k = k_y - k_x$, and where $\hat{V}$ acts on $[-1/2, 1/2]$ with kernel $\frac{i}{\pi} \theta'_1(0)$

computing these Fredholm determinants, we obtain the form factor in the Bethe basis

$$
\frac{1 - (-1)^k}{2} (-1)^{m-1} \prod_{m=1}^{+\infty} \frac{(1 - \bar{q}^m)^2(1 + \bar{p}^m \bar{q}^{-m})^2}{(1 + \bar{q}^m)^2(1 - \bar{p}^m \bar{q}^{-m})^2} \\
\times \lim_{\alpha \to 0} \left\{ \frac{i}{\pi(L-r)} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} e^{2\pi i \left( \frac{r+2\ell}{2(L-r)} + \eta \alpha \right)s} \frac{\theta_1(\bar{\eta}s + \frac{L+2\ell}{2(L-r)} + \alpha)}{\theta_1(\bar{\eta}s)} \theta_1' \left( \frac{L+2\ell}{2(L-r)} + \alpha \right) \right\} + O(N^{-\infty}).
$$

The parameter $\alpha$ is here to regularize the formula

this form factor depends only on the difference $k = k_y - k_x$ and $\ell = \ell_y - \ell_x$ of the quantum numbers.

If $k = 0$, the form factor vanishes $\sim \Bethe$ basis is not polarized.
Spontaneous staggered polarization

- A polarized basis is, for \( \epsilon \in \mathbb{Z}/2\mathbb{Z} \) and \( t \in \mathbb{Z}/(L - r)\mathbb{Z} \),

\[
|\epsilon, t\rangle = \frac{1}{\sqrt{2(L - r)}} \sum_{k=0}^{L-r-1} \sum_{\ell=0}^{L-r-1} (-1)^{k\epsilon} e^{-i\pi \frac{rk + 2\ell}{L-r}(t+s_0)} |k, \ell\rangle \langle k, \ell| \frac{1}{2(L-r)}
\]

(2) tends to the flat ground state configuration \((t, t+1, t, t+1, \ldots)\) or \((t+1, t, t+1, t, \ldots)\) in the low temperature limit \((\tau \to 0)\)

- In this basis, the form factor is diagonal (spontaneous polarization)

\[
\langle \epsilon, t | \sigma^z_m | \epsilon', t' \rangle = \delta_{\epsilon, \epsilon'} \delta_{t, t'} (-1)^{m+1} \epsilon \prod_{m=1}^{+\infty} \frac{(1 - \bar{q}^m)(1 - \bar{p}^m\bar{q}^{-m-t})}{(1 + \bar{q}^m)^2 (1 + \bar{p}^m\bar{q}^{-m-t})}
\]

\[
\prod_{m=0}^{+\infty} \frac{(1 - \bar{p}^m\bar{q}^{-m+t})}{(1 + \bar{p}^m\bar{q}^{-m+t})} + O(N^{-\infty}).
\]

with \( \bar{p} = e^{2i\pi \bar{\tau}}, \bar{q} = e^{2i\pi \bar{\eta}} \) and \( s_0 = \frac{1}{2\bar{\eta}} \).

- With conjugate periods, it is equal to

\[
\langle \epsilon, t | \sigma^z_m | \epsilon, t \rangle = (-1)^{m+1} \epsilon \frac{i\tau}{\pi\eta} \frac{\theta'_1(0; \frac{\tau}{\eta})}{\theta'_4(0; \frac{\tau}{\eta})} \frac{\theta_1(\frac{\eta t}{1-\eta}; \frac{\eta}{1-\eta})}{\theta_4(\frac{\eta t}{1-\eta}; \frac{\eta}{1-\eta})} + O(N^{-\infty}).
\]

Multi-point Local Height Probabilities

- For $|\epsilon, t\rangle$, one of the $2(L - r)$ ground states of the CSOS model compatible with the flat configurations
- The multi-point local height probabilities are defined as the thermodynamic limit of,

$$\tilde{P}_{\alpha_1, \ldots, \alpha_m}(s; \epsilon, t) = \langle \epsilon, t | \delta_s E_1^{\alpha_1, \alpha_1} \ldots, E_m^{\alpha_m, \alpha_m} | \epsilon, t \rangle$$  \hspace{1cm} (3)

for $\alpha_j \in \{+,-\}$, $\delta_s(s) = \delta_{s,s}$.
- Probability that on the same vertical line of the lattice, the height takes the successive values $s, s + \alpha_1, \ldots, s + \alpha_1 + \ldots + \alpha_m$.
- (3) can be computed directly from the multi-point matrix elements

$$P_{\alpha_1, \ldots, \alpha_m}(s; \{u\}, \omega_u, \{v\}, \omega_v) = \frac{\langle \{u\}, \omega_u | \delta_s E_1^{\alpha_1, \alpha_1} \ldots, E_m^{\alpha_m, \alpha_m} | \{v\}, \omega_v \rangle}{\langle \{u\}, \omega_u | \{u\}, \omega_u \rangle^{1/2} \langle \{v\}, \omega_v | \{v\}, \omega_v \rangle^{1/2}}$$

- QIP expresses the elementary matrices as generators of the YB algebra
  $\rightsquigarrow$ acting on the right state, $P_{\alpha_1, \ldots, \alpha_m}$ is expressed as sum of determinants
  $\rightsquigarrow$ sums over Bethe roots and inhomogeneities related to the action of the Yang-Baxter algebra $\rightsquigarrow$ sum over all the values of the dynamical parameter ($L$ terms) related to the partial scalar product formula
- Simplest example: local height probability ($m = 0$)
We want to compute \( \bar{P}(s; \epsilon, t) = \langle \epsilon, t | \delta_s | \epsilon, t \rangle \).

Start from the one-point matrix elements of \(\delta_s\) in the Bethe basis
\[
\frac{\langle \{u\}, \omega_u | \delta_s | \{v\}, \omega_v \rangle \langle \{v\}, \omega_v | \{v\}, \omega_v \rangle}{\langle \{v\}, \omega_v | \{v\}, \omega_v \rangle} \left( \frac{\langle \{u\}, \omega_u | \{u\}, \omega_u \rangle}{\langle \{u\}, \omega_u | \{u\}, \omega_u \rangle} \right)^{1/2}
\]

Recall that scalar product is defined as
\[
\langle \{u\}, \omega_u | \{v\}, \omega_v \rangle = \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \bar{\phi}(s) \phi(s) S_n(\{u\}; \{v\}; s)
\]
with \(S_n(\{u\}; \{v\}; s)\) the partial scalar product
\[
S_n(\{u\}; \{v\}; s) \propto \sum_{\nu=0}^{L-1} q^{\nu s} a^{(\nu)}(s_0) \det_n \left[ \Omega^\nu(\{u\}; \{v\}) \right], \text{ with }
\]
\(\Omega^\nu(\{u\}; \{v\})\) the scalar product matrix deformed by \(q^{\pm \nu} = e^{\pm 2i \pi \nu \eta}\).

The one-point matrix element of \(\delta_s\) is expressed as a sum of \(L\) terms
\[
\frac{\langle \{u\}, \omega_u | \delta_s | \{v\}, \omega_v \rangle \langle \{v\}, \omega_v | \{v\}, \omega_v \rangle}{\langle \{v\}, \omega_v | \{v\}, \omega_v \rangle} \propto \prod_{j < k} \frac{|v_j - v_k|}{|u_j - u_k|} \frac{1}{L} \sum_{\nu=0}^{L-1} q^{\nu s} a^{(\nu)}(s_0) \frac{\det_n \left[ \Omega^\nu(\{u\}; \{v\}) \right]}{\det_n[\Phi(\{v\})]},
\]

Convenient to insert the algebraic factor inside the determinant in order to take the thermodynamic limit.
One-point local height probabilities

- Introduce the matrix \( \mathcal{X} \) to simplify the algebraic factor, with arbitrary \( \gamma \)

\[
[\mathcal{X}]_{jk} = \frac{[0]'}{\sum_{l=1}^n (u_l - v_l) + \gamma} \frac{\prod_{l=1}^n [u_k - v_l]}{\prod_{l \neq k} [u_k - u_l]} \frac{[v_j - u_k + \sum_{l=1}^n (u_l - v_l) + \gamma]}{[v_j - u_k]},
\]

such that

\[
\det_n [\mathcal{X}] = (-[0]')^n \frac{[\gamma]}{\sum_{l=1}^n (u_l - v_l) + \gamma} \prod_{j < k} \frac{[v_j - v_k]}{[u_j - u_k]},
\]

- Determinant identity

\[
\prod_{j < k} \frac{[v_j - v_k]}{[u_j - u_k]} \det_n [\Omega^{(\nu)}] \propto \det_n [\mathcal{X} \Omega^{(\nu)}] \propto \det_n [\mathcal{H}^{(\nu)}]
\]

- When \( \{\nu\}, \omega_\nu \to \{u\}, \omega_u \), \( \mathcal{H}^{(\nu)} \to \Phi \) for \( \nu \neq 0 \)

- At the thermodynamic limit, for two ground states, the last determinant with \( n = \frac{N}{2} \) tends to a Fredholm determinant

\[
\det_n [\mathcal{H}^{(\nu)}] \to \det [1 + \hat{K}^{(\eta(\tilde{\gamma} - \nu) + |x| - |y|)}] + O(N^{-\infty})
\]

where \( \hat{K}^{(\gamma)} \) acts on \([ -\frac{1}{2}, \frac{1}{2} ]\) with kernel \( K^{(\gamma)}_X \) given by

\[
K^{(\gamma)}_X(z) = \frac{i}{2\pi} \frac{\theta'_1(0)}{\theta_1(X)} \left\{ e^{2i\pi \gamma} \frac{\theta_1(z + X + \tilde{\eta})}{\theta_1(z + \tilde{\eta})} - e^{-2i\pi \gamma} \frac{\theta_1(z + X - \tilde{\eta})}{\theta_1(z - \tilde{\eta})} \right\}
\]
One-point local height probabilities

- **Bethe basis:** matrix elements of $\delta_s$ at the thermodynamic limit

$$P(s; k_x, \ell_x; k_y, \ell_y) \propto e^{-i\pi s \left(-\frac{r k + 2\ell}{L - r} + 2\eta \tilde{\gamma}\right)} f_{\tilde{\gamma}}(k, \ell, s) \sum_{\nu=0}^{L-1} q^{\nu s} a_{\tilde{\gamma}}^{(\nu)}(s_0) g_{\tilde{\gamma}}(k, \ell)$$

with $k = k_y - k_x$, $\ell = \ell_y - \ell_x$, $f_{\tilde{\gamma}}(k, \ell, s)$ and $g_{\tilde{\gamma}}(k, \ell)$ ratio of $\theta$ functions.

- **Polarized basis:** Local height probability is diagonal

- for even L:
  $$\sim \epsilon + s - t \text{ odd, } \bar{P}(s; \epsilon, t) = \langle \epsilon, t | \delta_s | \epsilon, t \rangle = 0$$
  $$\sim \epsilon + s - t \text{ even, }$$

$$\bar{P}(s; \epsilon, t) = \frac{2}{L} \frac{\theta_4(\frac{r}{L} \tilde{s}; \tau)}{\theta_4(\eta \tilde{t} \frac{L}{L-r}; \frac{L}{L-r} \tau) \theta_4(0; \frac{r}{L} \frac{L}{L-r} \tau)} \theta_3\left(-\frac{\tilde{t}}{L-r} + \frac{\tilde{s}}{L} ; \frac{r}{L-r} \right) + O(\tilde{N}^{-\infty})$$

- for odd L

$$\bar{P}(s; \epsilon, t) = \frac{1}{L} \frac{\theta_4(\frac{r}{L} \tilde{s}; \tau) \theta_3\left(\frac{\tilde{s}}{2L} - \frac{\tilde{t}}{2(L-r)} + \frac{\tilde{s} - \epsilon}{2} ; \frac{r}{4(L-r)} \right)}{\theta_4(0; \frac{r}{L} \tau) \theta_4(\frac{r \tilde{t}}{L-r}; \frac{L \tau}{L-r})} + O(\tilde{N}^{-\infty})$$

with $\tilde{s} = s - \frac{1}{2\tilde{\eta}}$, $\tilde{t} = t - \frac{1}{2\tilde{\eta}}$.

$$\sim$$ same expressions as Pearce & Seaton
Multi-point matrix elements

Multi-point matrix elements in finite volume

\[ P_{\alpha_1, \ldots, \alpha_m} = \frac{\langle \{ u \}, \omega_u | \delta_s E_1^{\alpha_1, \alpha_1} \ldots, E_m^{\alpha_m, \alpha_m} | \{ v \}, \omega_v \rangle}{\langle \{ v \}, \omega_v | \{ v \}, \omega_v \rangle} \cdot \left( \frac{\langle \{ u \}, \omega_u | \{ u \}, \omega_u \rangle}{\langle \{ v \}, \omega_v | \{ v \}, \omega_v \rangle} \right)^{1/2} \]

QIP expresses the elementary matrices as generators of the YB algebra \( \rightsquigarrow \) acting on the right state, \( P_{\alpha_1, \ldots, \alpha_m} \) is expressed as sum of determinants (commutation relations + partial scalar products)

\[ P_{\alpha_1, \ldots, \alpha_m} \propto \sum_{\{ b_p \}} G_{\alpha_1, \ldots, \alpha_m}^\gamma (s; \{ v_{bp} \}, \{ \xi \}) \times \sum_{\nu=0}^{L-1} q^\nu s a^{(\nu)}(s_0) \frac{\det_n[\mathcal{H}^{(\nu)}_{\gamma; \{ b_p \}}]}{\det_n[\Phi(\{ v \})]} , \]

\( G_{\alpha_1, \ldots, \alpha_m}^\gamma \) admits a similar algebraic part that the elementary blocks of the XXZ chain + a dynamical part

\( N - m \) columns of \( \mathcal{H}^{(\nu)}_{\gamma; \{ b_p \}} \) are of the form of the 1-point matrix elements determinant \( \mathcal{H}^{(\nu)}_{\gamma} \)

\( m \) columns of \( \mathcal{H}^{(\nu)}_{\gamma; \{ b_p \}} \) are of “form factor type”
Multi-point matrix elements

- How to take the thermodynamic limit?

\[
\frac{\det_n [\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}]}{\det_n [\Phi]} = \frac{\det_n [\mathcal{H}^{(\nu)}_{\gamma}]}{\det_n [\Phi]} \underbrace{\det_n [\mathcal{H}^{(\nu)}_{\gamma}^{-1} \mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}]}_{\mathcal{A}}
\underbrace{\det_n [\Phi]}_{\mathcal{B}}
\]

- At the thermodynamic limit

\(\mathcal{A}\) is equal to the 1-point matrix elements determinants, which tends to two Fredholm determinants we already computed

\(\mathcal{B} = \det_m \left[ S^{(\nu)}_{\gamma;\{b_p\}} \right]\) has a finite size \(m\) (length of the correlation function)

- Matrix elements of \(S^{(\nu)}_{\gamma;\{b_p\}}\) can be computed explicitly at the thermodynamic limit and are expressed with a “modified” density \(\rho^{(\nu)}_{\gamma}\)

- Finally,

\[
\langle k_x, \ell_x | \delta_s E_1^{\alpha_1 \alpha_1} \ldots E_m^{\alpha_m \alpha_m} | k_y, \ell_y \rangle = \sum_{\{b_p\}} \tilde{G}_{\alpha_1,\ldots,\alpha_m}(s; \{y_{b_p}\}, \{\zeta\})
\]

\[
\times \sum_{\nu=0}^{L-1} q^{\nu s} a^{(\nu)}(s_0) \frac{\det_n [\tilde{\mathcal{H}}^{(\nu)}_{\gamma}(\{x\}, \omega_x; \{y\}, \omega_y)]}{\det_n [\tilde{\Phi}(\{y\})]} \det_m [\tilde{S}^{(\nu)}_{\gamma;\{b_p\}}]
\]
Multi-point matrix elements

Extracting a 1-point matrix element parts from $\tilde{S}_\gamma^{(\nu)}; \{b_p\}$

Sums over Bethe roots become integrals such that in the Bethe basis, the multi-point matrix elements looks like (with $|z| - |\zeta| = \sum_{t=1}^{m} z_t - \zeta_t$)

$$
\mathbb{P}_{\alpha_1, \ldots, \alpha_m}(s; k_x, \ell_x; k_y, \ell_y) \propto \int_{C_-} \prod_{j=1}^{\lfloor \alpha_\cdot \rfloor} dz_j \int_{C_+} \prod_{j=\lfloor \alpha_\cdot \rfloor + 1}^{m} dz_j \widetilde{G}_{\alpha_1, \ldots, \alpha_m}(s; \{z\}, \{\zeta\})
$$

$$
\times \tilde{S}_m(\{z\}; \{\zeta\}) \underbrace{\mathbb{P}(s, |z| - |\zeta|; k, \ell)}_{\text{determinant contribution}} + \underbrace{\mathbb{P}(s, |z| - |\zeta|; k, \ell)}_{\text{algebraic part}} + O(N^{-\infty}), \quad (4)
$$

Integration contours are such that $C_- = [-1/2, 1/2]$, $C_+ = C_- \cup \Gamma(\{\xi\})$.

ground states dependance contained inside the deformed 1-point M.E. $\mathbb{P}(s, |z| - |\zeta|; k, \ell)$ (analytical part). The latter is such that

$$
\mathbb{P}(s; k_x, \ell_x; k_y, \ell_y) = \mathbb{P}(s, 0; k, \ell) + O(N^{-\infty})
$$

Representation of (4) is similar to the elementary blocks of the XXZ chain

The sum over the dynamical parameter ($L$ terms) entirely contained inside $\mathbb{P}(s, |z| - |\zeta|; k, \ell)$
Multi-point local height probabilities

- **Polarized basis:**
  \[ \bar{P}(s, Z; \epsilon, t) = \sum_{k=0}^{1} \sum_{\ell=0}^{L-r-1} (-1)^k \epsilon \ e^{-i\pi \frac{rk+2\ell}{L-r}(t+s_0)} \bar{P}(s, Z; k, \ell) \]
  
  only the analytical part is changed (resummation is possible)
  \[ \bar{P}_{\alpha_1,\ldots,\alpha_m}(s; \epsilon, t) \propto \int_{C_-} \prod_{j=1}^{m} dz_j \int_{C_+} \prod_{j=|\alpha_-|+1}^{m} dz_j \tilde{G}_{\alpha_1,\ldots,\alpha_m}(s; \{z\}, \{\zeta\}) \]
  \[ \times \tilde{S}_m(\{z\}; \{\zeta\}) \bar{P}(s, |z| - |\zeta|; \epsilon, t) + O(N^{-\infty}). \]

- **MPLHP are diagonals:** ground states sectors are frozen
  \[ \bar{P}(s, Z; \epsilon, t) \bigg|_{L \text{ even}} = 0, \quad \text{with } \tilde{s}_0 = s_0 + \frac{1}{2\tilde{\eta}} \in \mathbb{R} \]
  \[ \bar{P}(s, Z; \epsilon, t) \bigg|_{L \text{ even}} = 2e^{i\pi \left(2\frac{L}{L} \tilde{s}Z - \frac{L-r}{L} Z^2 \tau \right)} \frac{\theta_4(\frac{r\tilde{s}}{L}; \tau) \theta_3(\tilde{s}_0 + t - \tilde{s} + Z \tau; r(\tilde{s}_0 + t - \frac{1}{2} Z \tau)) \theta_4(0; \frac{L}{r} \tau) \theta_4 \left( \frac{r(\tilde{s}_0 + t)}{L-r}; \frac{L}{L-r} \tau \right)}{L \theta_4(0; \frac{L}{r} \tau) \theta_4 \left( \frac{r(\tilde{s}_0 + t)}{L-r}; \frac{L}{L-r} \tau \right)}, \]

  \[ \bar{P}(s, Z; \epsilon, t) \bigg|_{L \text{ odd}} = e^{i\pi \left(2\frac{L}{L} \tilde{s}Z - \frac{L-r}{L} Z^2 \tau \right)} \frac{\theta_4(\frac{r\tilde{s}}{L}; \tau) \theta_3 \left( \frac{1}{2} - \frac{1}{2L} \tilde{s} - \frac{1}{2} \frac{1}{2(L-r)} (\tilde{s}_0 + t) - \frac{\epsilon}{2} + \frac{Z}{2r} \tau; \frac{r(\tilde{s}_0 + t)}{L-r}, \frac{L}{L-r} \tau \right)}{L \theta_4(0; \frac{L}{r} \tau) \theta_4 \left( \frac{r(\tilde{s}_0 + t)}{L-r}; \frac{L}{L-r} \tau \right)}, \]

- single \( m \)-fold integral as for the XXZ spin chain
Conclusion and perspectives

- **Summary of the results obtained for the cyclic SOS model**
  - Determinant representations for scalar products/norms of Bethe states/form factors in finite volume (Véronique’s talk)
  - Representation as $L$ multiple sums of determinants for the Multi-point matrix elements in finite volume
  - Study of the thermodynamic limit:
    - $\Rightarrow$ Explicit result for the spontaneous polarization at the thermodynamic limit
    - $\Rightarrow$ Single $m$-fold integral formula for the MPLHP

- **Further questions . . .**
  - Study of two-point correlation functions in the thermodynamic limit
  - Unrestricted SOS model? ($\eta \in \mathbb{R}$)
  - **XYZ model?**
    - $\Rightarrow$ Combinatorial complexity of Vertex-IRF transformation