Correlation functions of the cyclic SOS model from algebraic Bethe Ansatz

Thermodynamic limit

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Articles:

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Computation of correlation functions within ABA

For $|\,\psi_g\,\rangle$ a ground state of the cyclic SOS model for which $\eta = \frac{r}{L}$,

$\langle\,\psi_g\,|\mathcal{O}|\,\psi_g\,\rangle \qquad \mathcal{O} = \delta_{\mathbf{s}},\ \sigma^z_m,\ \delta_{\mathbf{s}} E_1^{\alpha_1\alpha_1}\dots E_m^{\alpha_m\alpha_m},\ \dots$

1. Diagonalization achieved by ABA Felder, Varchenko (1996) :

   $$|\,\{u\},\omega\,\rangle : s \mapsto \varphi(s)B(u_1;s)B(u_2;s-1)\dots B(u_n;s-n+1)|\,0\,\rangle \in \mathrm{Fun}(\mathcal{H}[0])$$

   for which $\varphi(s) = \omega^s \prod_{j=1}^n \frac{[1]}{[s-j]}$ and with $\{u_1,\dots,u_n\}$, solution of

   $$a(u_j)\prod_{l\neq j}\frac{[u_l-u_j+1]}{[u_l-u_j]} = (-1)^{rk}\omega^{-2}\,d(u_j)\prod_{l\neq j}\frac{[u_j-u_l+1]}{[u_j-u_l]}, \quad j=1,\dots,n,$$

   with $N = 2n+kL$ ($k\in\mathbb{Z}$) and $\omega^L = (-1)^{rn}$ (for $\eta = r/L$).
2. To characterize $|\,\psi_g\,\rangle$ $\rightsquigarrow$ this talk
3. Quantum inverse problem solved similarly as for the XXZ spins chain
4. Single determinant representation for the scalar product between a Bethe eigenstate and a Bethe arbitrary state (in particular for the norm, and for the form factors)

   $\rightsquigarrow$ partial scalar product expressed as a sum of L terms because of the dynamical parameter
5. To take the thermodynamic limit ($N\to\infty$) $\rightsquigarrow$ this talk

Ground states and thermodynamic limit

Ground states:

It is more convenient to work with conjugate periods $\tilde{\eta} = -\frac{\eta}{\tau}$, $\tilde{\tau} = -\frac{1}{\tau}$

- Jacobi's imaginary transformation: $[u] \propto e^{i\pi\eta\tilde{\eta}u^2}\theta_1(x = \tilde{\eta}u; \tilde{\tau})$
- Logarithmic Bethe ansatz equations for $(\{x_j\}_{1\leq j\leq n}, \omega_x = exp(i\pi\frac{rn+2\ell_x}{L}))$:

$$Np_0(x_j) - \sum_{l=1}^{n} \vartheta(x_j - x_l) = 2\pi\left(n_j - \frac{n+1}{2} + \frac{rn+2\ell_x}{L} + 2\eta\sum_{l=1}^{n} x_l\right), 1 \leq j \leq n$$

  with $p_0(z) = i\log\frac{\theta_1(\tilde{\eta}/2+z)}{\theta_1(\tilde{\eta}/2-z)}$ (bare momentum), $\vartheta(z) = i\log\frac{\theta_1(\tilde{\eta}+z)}{\theta_1(\tilde{\eta}-z)}$ (bare phase), $n_j \in \mathbb{Z}$.
- When η is rational ($\eta = r/L$), in the domain $0 < \eta < \frac{1}{2}$ and for $N \to +\infty$, all roots x_j of the ground states, for which $n = N/2$ and $n_{j+1} - n_j = 1$, are real and densely fill $[-1/2; 1/2]$ with density $\rho(x)$.
- In the thermodynamic limit: $2(L - r)$ degenerate Bethe ground states, labelled by quantum numbers $k \in \mathbb{Z}/2\mathbb{Z}$ and $\ell \in \mathbb{Z}/(L-r)\mathbb{Z}$ (Pearce, Batchelor).
- We now consider two ground states $|\,\{x = \tilde{\eta}u\}, \omega_x\,\rangle = |\,\mathrm{k}_x, \ell_x\,\rangle$, $|\,\{y = \tilde{\eta}v\}, \omega_y\,\rangle = |\,\mathrm{k}_y, \ell_y\,\rangle$
- Sum rules for roots of Bethe ground states?

Finite size corrections

Let f be a 1-periodic and $\mathcal{C}^\infty(\mathbb{R})$ function. If $\{x_j\}_{1\leq j\leq n}$ parametrizes one of the $2(L-r)$ ground states,

$$\frac{1}{N}\sum_{j=1}^{n} f(x_j) = \int_{-1/2}^{1/2} f(z)\rho(z)dz + O(N^{-\infty}).$$

If g is $\mathcal{C}^\infty(\mathbb{R})$ function such that g' is 1-periodic,

$$\frac{1}{N}\sum_{j=1}^{n} g(x_j) = \int_{-1/2}^{1/2} g(z)\rho(z)dz + \frac{c_g}{N}\sum_{j=1}^{n} x_j + O(N^{-\infty}),$$

where $c_g = g(1/2) - g(-1/2)$.

$\leadsto$ this allows us to compute the sum rules while controlling finite size corrections

- $(\{x_j\}, \mathrm{k}_x, \ell_x)$ and $(\{y_k\}, \mathrm{k}_y, \ell_y)$ two Bethe ground states

$$\sum_{l=1}^{n}(x_l - y_l) = \frac{L(\mathrm{k}_x - \mathrm{k}_y) + 2(\ell_x - \ell_y)}{2(L-r)} + O(N^{-\infty}),$$

with $\mathrm{k}_x, \mathrm{k}_y \in \mathbb{Z}/2\mathbb{Z}$, and $\ell_x, \ell_y \in \mathbb{Z}/(L-r)\mathbb{Z}$.

Thermodynamic limit of the form factor

- we want to compute the form factor at the thermodynamic limit. between two ground states

$$\frac{\langle\, k_x,\ell_x \,|\, \sigma^z_m \,|\, k_y,\ell_y \,\rangle}{(\langle\, k_x,\ell_x \mid k_x,\ell_x \,\rangle\langle\, k_y,\ell_y \mid k_y,\ell_y \,\rangle)^{1/2}} = \underbrace{\frac{\langle\, k_x,\ell_x \,|\, \sigma^z_m \,|\, k_y,\ell_y \,\rangle}{\langle\, k_y,\ell_y \mid k_y,\ell_y \,\rangle}}_{\mathcal{M}} \cdot \underbrace{\left(\frac{\langle\, k_y,\ell_y \mid k_y,\ell_y \,\rangle}{\langle\, k_x,\ell_x \mid k_x,\ell_x \,\rangle}\right)^{1/2}}_{\mathcal{N}}$$

- The norm is expressed as a single determinant representation of size n, $\mathcal{N}^2 \propto \frac{\det_n\left[\widetilde{\Phi}(\{y\})\right]}{\det_n\left[\widetilde{\Phi}(\{x\})\right]}$
- at the thermodynamic limit, the determinant with $n = N/2$ tends to Fredholm determinants

$$\det_n\left[\widetilde{\Phi}(\{y\})\right] = (-2\pi i\tilde{\eta}N)^n \prod_{l=1}^{n} \rho(y_l)\left\{\det\left[1+\widehat{K}-\widehat{V}_0\right] + O(N^{-\infty})\right\}$$

  with integral operators $\widehat{K}$ and $\widehat{V}_0$ acting on $[-1/2, 1/2]$ with respective kernel $K(y-z) = \frac{i}{2\pi}\left\{\frac{\theta_1'(y-z+\tilde{\eta})}{\theta_1(y-z+\tilde{\eta})} - \frac{\theta_1'(y-z-\tilde{\eta})}{\theta_1(y-z-\tilde{\eta})}\right\}$ and $V_0(y-z) = 2\eta$
- We eventually find $\mathcal{N}^2 = \left(\frac{\omega_y}{\omega_x}\right)^{2n} = \left(e^{2i\pi\frac{\ell_y-\ell_x}{L}}\right)^{2n}$
- $\langle\, k_x,\ell_x \,|\, \sigma^z_m \,|\, k_y,\ell_y \,\rangle$ expressed as a difference of determinants of size n

Thermodynamic limit of the form factor

- Similarly, $\mathcal{M}$ is expressed as a ratio of Fredholm determinants,

$$\mathcal{M} \propto \frac{\det\left[1 + (-1)^{\mathrm{k}}\widehat{K} + \frac{1-(-1)^{\mathrm{k}}}{2}\widehat{V}\right] - \det\left[1 + (-1)^{\mathrm{k}}\widehat{K} - \frac{1-(-1)^{\mathrm{k}}}{2}\widehat{V}\right]}{\det\left[1 + \widehat{K} - \widehat{V}_0\right]} + O(N^{-\infty})$$

  with $\mathrm{k} = \mathrm{k}_y - \mathrm{k}_x$, and where $\widehat{V}$ acts on $[-1/2, 1/2]$ with kernel $\frac{i}{\pi}\theta_1'(0)$
- computing these Fredholm determinants, we obtain the form factor in the Bethe basis

$$\frac{1-(-1)^{\mathrm{k}}}{2}(-1)^{m-1}\prod_{m=1}^{+\infty}\frac{(1-\tilde{q}^m)^2(1+\tilde{p}^m\tilde{q}^{-m})^2}{(1+\tilde{q}^m)^2(1-\tilde{p}^m\tilde{q}^{-m})^2}$$

$$\times \lim_{\alpha\to 0}\left\{\frac{i}{\pi(L-r)}\sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} e^{2\pi i(\frac{r+2\ell}{2(L-r)}+\eta\alpha)s}\frac{\theta_1\left(\tilde{\eta}s + \frac{L+2\ell}{2(L-r)} + \alpha\right)\theta_1'(0)}{\theta_1(\tilde{\eta}s)\,\theta_1\left(\frac{L+2\ell}{2(L-r)} + \alpha\right)}\right\} + O(N^{-\infty}).$$

- The parameter α is here to regularize the formula
- this form factor depends only on the difference $\mathrm{k} = \mathrm{k}_y - \mathrm{k}_x$ and $\ell = \ell_y - \ell_x$ of the quantum numbers.
- If $\mathrm{k} = 0$, the form factor vanishes $\rightsquigarrow$ Bethe basis is not polarized.

Spontaneous staggered polarization

- a polarized basis is, for $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{t} \in \mathbb{Z}/(L-r)\mathbb{Z}$,

$$|\,\epsilon, \mathrm{t}\,\rangle = \frac{1}{\sqrt{2(L-r)}} \sum_{\mathrm{k}=0}^{1} \sum_{\ell=0}^{L-r-1} (-1)^{k\epsilon} e^{-i\pi \frac{r\mathrm{k}+2\ell}{L-r}(\mathrm{t}+s_0)} \frac{|\,\mathrm{k}, \ell\,\rangle}{\langle\,\mathrm{k}, \ell\,|\,\mathrm{k}, \ell\,\rangle^{1/2}} \qquad (2)$$

(2) tends to the flat ground state configuration $(\mathrm{t}, \mathrm{t}+1, \mathrm{t}, \mathrm{t}+1, \ldots)$ or $(\mathrm{t}+1, \mathrm{t}, \mathrm{t}+1, \mathrm{t}, \ldots)$ in the low temperature limit $(\tau \to 0)$

- In this basis, the form factor is diagonal (spontaneous polarization)

$$\langle\,\epsilon, \mathrm{t}\,|\,\sigma_m^z\,|\,\epsilon', \mathrm{t}'\,\rangle = \delta_{\epsilon,\epsilon'}\delta_{\mathrm{t},\mathrm{t}'}(-1)^{m-1+\epsilon} \prod_{m=1}^{+\infty} \frac{(1-\tilde{q}^m)^2\,(1-\tilde{p}^m\tilde{q}^{-m-\mathrm{t}})}{(1+\tilde{q}^m)^2\,(1+\tilde{p}^m\tilde{q}^{-m-\mathrm{t}})}$$

$$\prod_{m=0}^{+\infty} \frac{(1-\tilde{p}^m\tilde{q}^{-m+\mathrm{t}})}{(1+\tilde{p}^m\tilde{q}^{-m+\mathrm{t}})} + O(N^{-\infty}).$$

with $\tilde{p} = e^{2i\pi\tilde{\tau}}$, $\tilde{q} = e^{2i\pi\tilde{\eta}}$ and $s_0 = \frac{1}{2\tilde{\eta}}$.

- with conjugate periods, it is equal to

$$\langle\,\epsilon, \mathrm{t}\,|\,\sigma_m^z\,|\,\epsilon, \mathrm{t}\,\rangle = (-1)^{m+\epsilon} \frac{i\tau}{\pi\eta} \frac{\theta_1'\left(0; \frac{\tau}{\eta}\right)\theta_1\left(\frac{\eta\,\mathrm{t}}{1-\eta}; \frac{\tau}{1-\eta}\right)}{\theta_4\left(0; \frac{\tau}{\eta}\right)\theta_4\left(\frac{\eta\,\mathrm{t}}{1-\eta}; \frac{\tau}{1-\eta}\right)} + O(N^{-\infty}).$$

⇝ Proof of a conjecture by Date & al *1990 J. Phys. A: Math. Gen. 23 L163*.

Multi-point Local Height Probabilities

- For $|\,\epsilon, \mathrm{t}\,\rangle$, one of the $2(L-r)$ ground states of the CSOS model compatible with the flat configurations
- The multi-point local height probabilities are defined as the thermodynamic limit of,

$$\bar{\mathbf{P}}_{\alpha_1,\ldots,\alpha_m}(\mathbf{s};\epsilon,\mathrm{t}) = \langle\,\epsilon,\mathrm{t}\,|\delta_{\mathbf{s}} E_1^{\alpha_1,\alpha_1}\ldots, E_m^{\alpha_m,\alpha_m}|\,\epsilon,\mathrm{t}\,\rangle \qquad (3)$$

  for $\alpha_j \in \{+,-\}$, $\delta_{\mathbf{s}}(s) = \delta_{\mathbf{s},s}$.
- Probability that on the same vertical line of the lattice, the height takes the successive values $\mathbf{s}$, $\mathbf{s}+\alpha_1$, ..., $\mathbf{s}+\alpha_1+\ldots+\alpha_m$.
- (3) can be computed directly from the multi-point matrix elements

$$\mathbb{P}_{\alpha_1,\ldots,\alpha_m}(\mathbf{s};\{u\},\omega_u,\{v\},\omega_v) = \frac{\langle\,\{u\},\omega_u\,|\delta_{\mathbf{s}} E_1^{\alpha_1,\alpha_1}\ldots, E_m^{\alpha_m,\alpha_m}|\,\{v\},\omega_v\,\rangle}{\langle\,\{u\},\omega_u\mid\{u\},\omega_u\,\rangle^{1/2}\langle\,\{v\},\omega_v\mid\{v\},\omega_v\,\rangle^{1/2}}$$

- QIP expresses the elementary matrices as generators of the YB algebra
  $\rightsquigarrow$ acting on the right state, $\mathbb{P}_{\alpha_1,\ldots,\alpha_m}$ is expressed as sum of determinants
  $\rightsquigarrow$ sums over Bethe roots and inhomogeneities related to the action of the Yang-Baxter algebra $\rightsquigarrow$ sum over all the values of the dynamical parameter (L terms) related to the partial scalar product formula
- Simplest example: local height probability ($m=0$)

One-point local height probabilities

- We want to compute $\bar{\mathbf{P}}(\mathbf{s};\epsilon,\mathrm{t}) = \langle\, \epsilon,\mathrm{t}\,|\delta_{\mathbf{s}}|\,\epsilon,\mathrm{t}\,\rangle$
- Start from the one-point matrix elements of $\delta_{\mathbf{s}}$ in the Bethe basis
$$\frac{\langle\,\{u\},\omega_u\,|\delta_{\mathbf{s}}|\,\{v\},\omega_v\,\rangle}{\langle\,\{v\},\omega_v\mid\{v\},\omega_v\,\rangle}\cdot\left(\frac{\langle\,\{v\},\omega_v\mid\{v\},\omega_v\,\rangle}{\langle\,\{u\},\omega_u\mid\{u\},\omega_u\,\rangle}\right)^{1/2}$$
- Recall that scalar product is defined as
$$\langle\{u\},\omega_u\,|\,\{v\},\omega_v\rangle = \frac{1}{L}\sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \bar{\varphi}(s)\,\varphi(s)\,S_n(\{u\};\{v\};s)$$
with $S_n(\{u\};\{v\};s)$ the partial scalar product
$$S_n(\{u\};\{v\};s) \propto \sum_{\nu=0}^{L-1} q^{\nu s} a_\gamma^{(\nu)}(s_0)\ \mathrm{det}_n\left[\Omega_\gamma^{(\nu)}(\{u\};\{v\})\right],$$ with
$\Omega_\gamma^{(\nu)}(\{u\};\{v\})$ the scalar product matrix deformed by $q^{\pm\nu} = e^{\pm 2i\pi\nu\eta}$
- The one-point matrix element of $\delta_{\mathbf{s}}$ is expressed as a sum of L terms
$$\frac{\langle\,\{u\},\omega_u\,|\,\delta_{\mathbf{s}}\,|\,\{v\},\omega_v\,\rangle}{\langle\{v\},\omega_v\mid\{v\},\omega_v\rangle} \propto \prod_{j<k}\frac{[v_j-v_k]}{[u_j-u_k]}\,\frac{1}{L}\sum_{\nu=0}^{L-1} q^{\nu s}\,a_\gamma^{(\nu)}(s_0)\,\frac{\mathrm{det}_n\left[\Omega_\gamma^{(\nu)}(\{u\};\{v\})\right]}{\mathrm{det}_n[\Phi(\{v\})]},$$
- Convenient to insert the algebraic factor inside the determinant in order to take the thermodynamic limit

One-point local height probabilities

- Introduce the matrix $\mathcal{X}$ to simplify the algebraic factor, with arbitrary γ

$$[\mathcal{X}]_{jk} = \frac{[0]'}{[\sum_{l=1}^n (u_l - v_l) + \gamma]} \frac{\prod_{l=1}^n [u_k - v_l]}{\prod_{l\neq k}[u_k - u_l]} \frac{[v_j - u_k + \sum_{l=1}^n (u_l - v_l) + \gamma]}{[v_j - u_k]},$$

such that

$$\det_n [\mathcal{X}] = (-[0]')^n \frac{[\gamma]}{[\sum_{l=1}^n (u_l - v_l) + \gamma]} \prod_{j<k} \frac{[v_j - v_k]}{[u_j - u_k]},$$

- determinant identity

$$\prod_{j<k} \frac{[v_j - v_k]}{[u_j - u_k]} \det_n \left[\Omega_\gamma^{(\nu)}\right] \propto \det_n \left[\mathcal{X}\Omega_\gamma^{(\nu)}\right] \propto \det_n \left[\mathcal{H}_\gamma^{(\nu)}\right]$$

- when $(\{v\}, \omega_v) \to (\{u\}, \omega_u)$, $\mathcal{H}_\gamma^{(\nu)} \nrightarrow \Phi$ for $\nu \neq 0$
- At the thermodynamic limit, for two ground states, the last determinant with $n = \frac{N}{2}$ tends to a Fredholm determinant

$$\det_n \left[\mathcal{H}_\gamma^{(\nu)}\right] \to \det\left[1 + \widehat{K}_{\tilde{\gamma}+|x|-|y|}^{(\eta(\tilde{\gamma}-\nu)+|x|-|y|)}\right] + O(N^{-\infty})$$

where $\widehat{K}_X^{(Y)}$ acts on $[-\frac{1}{2}, \frac{1}{2}]$ with kernel $K_X^{(Y)}$ given by

$$K_X^{(Y)}(z) = \frac{i}{2\pi} \frac{\theta_1'(0)}{\theta_1(X)} \left\{ e^{2i\pi Y} \frac{\theta_1(z + X + \tilde{\eta})}{\theta_1(z + \tilde{\eta})} - e^{-2i\pi Y} \frac{\theta_1(z + X - \tilde{\eta})}{\theta_1(z - \tilde{\eta})} \right\}$$

One-point local height probabilities

- **Bethe basis:** matrix elements of $\delta_{\mathbf{s}}$ at the thermodynamic limit

$$\mathbb{P}(\mathbf{s};\mathbf{k}_x,\ell_x;\mathbf{k}_y,\ell_y) \propto e^{-i\pi\mathbf{s}\left(-\frac{r\mathrm{k}+2\ell}{L-r}+2\eta\tilde{\gamma}\right)} f_{\tilde{\gamma}}(\mathrm{k},\ell,\mathbf{s}) \sum_{\nu=0}^{L-1} q^{\nu\mathbf{s}}\ a_{\gamma}^{(\nu)}(s_0) g_{\tilde{\gamma}}(\mathrm{k},\ell)$$

  with $\mathrm{k}=\mathrm{k}_y-\mathrm{k}_x$, $\ell=\ell_y-\ell_x$, $f_{\tilde{\gamma}}(\mathrm{k},\ell,\mathbf{s})$ and $g_{\tilde{\gamma}}(\mathrm{k},\ell)$ ratio of θ functions.
  Polarized basis: Local height probability is diagonal

- for even L:
  ⇝ $\epsilon+\mathbf{s}-\mathrm{t}$ odd, $\bar{\mathbf{P}}(\mathbf{s};\epsilon,\mathrm{t}) = \langle\,\epsilon,\mathrm{t}\,|\,\delta_{\mathbf{s}}\,|\,\epsilon,\mathrm{t}\,\rangle = 0$
  ⇝ $\epsilon+\mathbf{s}-\mathrm{t}$ even,

$$\bar{\mathbf{P}}(\mathbf{s};\epsilon,\mathrm{t}) = \frac{2}{L}\frac{\theta_4(\frac{r}{L}\tilde{\mathbf{s}};\tau)}{\theta_4(\eta\tilde{\mathrm{t}}\frac{L}{L-r};\frac{L}{L-r}\tau)}\frac{\theta_3(-\frac{\tilde{\mathrm{t}}}{(L-r)}+\frac{\tilde{\mathbf{s}}}{L};\frac{\tau}{r(L-r)})}{\theta_4(0;\frac{L}{r}\tau)} + O(N^{-\infty})$$

- for odd L

$$\bar{\mathbf{P}}(\mathbf{s};\epsilon,\mathrm{t}) = \frac{1}{L}\frac{\theta_4(\frac{r}{L}\tilde{\mathbf{s}};\tau)\theta_3(\frac{\tilde{\mathbf{s}}}{2L}-\frac{\tilde{\mathrm{t}}}{2(L-r)}+\frac{\tilde{\mathrm{t}}-\tilde{\mathbf{s}}+\epsilon}{2};\frac{\tau}{4r(L-r)})}{\theta_4(0;\frac{L}{r}\tau)\theta_4(\frac{r\tilde{\mathrm{t}}_2}{L-r};\frac{L\tau}{L-r})} + O(N^{-\infty})$$

with $\tilde{\mathbf{s}}=\mathbf{s}-\frac{1}{2\tilde{\eta}}$, $\tilde{\mathrm{t}}=\mathrm{t}-\frac{1}{2\tilde{\eta}}$.
⇝ same expressions as Pearce & Seaton

Multi-point matrix elements

- Multi-point matrix elements in finite volume

$$\mathbb{P}_{\alpha_1,\ldots,\alpha_m} = \frac{\langle \{u\},\omega_u \,|\delta_s E_1^{\alpha_1,\alpha_1} \ldots, E_m^{\alpha_m,\alpha_m}|\, \{v\},\omega_v \rangle}{\langle \{v\},\omega_v \mid \{v\},\omega_v \rangle} \cdot \left(\frac{\langle \{v\},\omega_v \mid \{v\},\omega_v \rangle}{\langle \{u\},\omega_u \mid \{u\},\omega_u \rangle} \right)^{1/2}$$

- QIP expresses the elementary matrices as generators of the YB algebra $\rightsquigarrow$ acting on the right state, $\mathbb{P}_{\alpha_1,\ldots,\alpha_m}$ is expressed as sum of determinants (commutation relations + partial scalar products)

$$\mathbb{P}_{\alpha_1,\ldots,\alpha_m} \propto \sum_{\{b_p\}} G^{\gamma}_{\alpha_1,\ldots,\alpha_m}(s;\{v_{b_p}\},\{\xi\}) \times \sum_{\nu=0}^{L-1} q^{\nu s}\, a_{\gamma}^{(\nu)}(s_0)\, \frac{\det_n\left[\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}\right]}{\det_n[\Phi(\{v\})]},$$

- $G^{\gamma}_{\alpha_1,\ldots,\alpha_m}$ admits a similar algebraic part that the elementary blocks of the XXZ chain + a dynamical part
- $N-m$ columns of $\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}$ are of the form of the 1-point matrix elements determinant $\mathcal{H}^{(\nu)}_{\gamma}$
- m columns of $\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}$ are of "form factor type"

Multi-point matrix elements

- How to take the thermodynamic limit ?
- $$\frac{\det_n\left[\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}\right]}{\det_n\left[\Phi\right]} = \underbrace{\frac{\det_n\left[\mathcal{H}^{(\nu)}_{\gamma}\right]}{\det_n\left[\Phi\right]}}_{\mathcal{A}} \underbrace{\det_n\left[\mathcal{H}^{(\nu)}_{\gamma}{}^{-1}\mathcal{H}^{(\nu)}_{\gamma;\{b_p\}}\right]}_{\mathcal{B}}$$
- At the thermodynamic limit
  $\rightsquigarrow$ $\mathcal{A}$ is equal to the 1-point matrix elements determinants, which tends to two Fredholm determinants we already computed
  $\rightsquigarrow$ determinant $\mathcal{B} = \det_m\left[\mathcal{S}^{(\nu)}_{\gamma;\{b_p\}}\right]$ has a finite size m (length of the correlation function)
- Matrix elements of $\mathcal{S}^{(\nu)}_{\gamma;\{b_p\}}$ can be computed explicitly at the thermodynamic limit and are expressed with a "modified" density $\rho^{(\nu)}_{\gamma}$
- Finally,

$$\frac{\langle\, \mathsf{k}_x, \ell_x \,|\, \delta_{\mathsf{s}}\, E_1^{\alpha_1\alpha_1} \dots E_m^{\alpha_m\alpha_m} \,|\, \mathsf{k}_y, \ell_y \,\rangle}{\langle \mathsf{k}_y, \ell_y \mid \mathsf{k}_y, \ell_y \rangle} = \sum_{\{b_p\}} \widetilde{G}_{\alpha_1,\dots,\alpha_m}(\mathsf{s}; \{y_{b_p}\}, \{\zeta\})$$

$$\times \sum_{\nu=0}^{L-1} q^{\nu \mathsf{s}}\; a^{(\nu)}_{\gamma}(s_0)\; \frac{\det_n\left[\widetilde{\mathcal{H}}^{(\nu)}_{\gamma}(\{x\},\omega_x;\{y\},\omega_y)\right]}{\det_n\left[\widetilde{\Phi}(\{y\})\right]} \det_m\left[\widetilde{\mathcal{S}}^{(\nu)}_{\gamma;\{b_p\}}\right]$$

Multi-point matrix elements

- Extracting a 1-point matrix element parts from $\widetilde{\mathcal{S}}^{(\nu)}_{\gamma;\{b_p\}}$
- Sums over Bethe roots become integrals such that in the Bethe basis, the multi-point matrix elements looks like (with $|z|-|\zeta|=\sum_{t=1}^{m} z_t-\zeta_t$)

$$\mathbb{P}_{\alpha_1,\ldots,\alpha_m}(\mathbf{s};\mathrm{k}_x,\ell_x;\mathrm{k}_y,\ell_y)\propto\int_{\mathcal{C}_-}\prod_{j=1}^{|\boldsymbol{\alpha}_-|}dz_j\int_{\mathcal{C}_+}\prod_{j=|\boldsymbol{\alpha}_-|+1}^{m}dz_j\,\underbrace{\widetilde{G}_{\alpha_1,\ldots,\alpha_m}(\mathbf{s};\{z\},\{\zeta\})}_{\text{algebraic part}}$$
$$\times\underbrace{\bar{\mathcal{S}}_m(\{z\};\{\zeta\})}_{\text{determinant contribution}}\;\underbrace{\bar{\mathbb{P}}(\mathbf{s},|z|-|\zeta|;\mathrm{k},\ell)}_{\text{deformed 1-point M.E.}}+O(N^{-\infty}),\quad(4)$$

- Integration contours are such that $\mathcal{C}_-=[-1/2,1/2]$, $\mathcal{C}_+=\mathcal{C}_-\cup\Gamma(\{\xi\})$,
- ground states dependance contained inside the deformed 1-point M.E. $\bar{\mathbb{P}}(\mathbf{s},|z|-|\zeta|;\mathrm{k},\ell)$ (analytical part). The latter is such that $\mathbb{P}(s;\mathrm{k}_x,\ell_x;\mathrm{k}_y,\ell_y)=\bar{\mathbb{P}}(s,0;\mathrm{k},\ell)+O(N^{-\infty})$
- Representation of (4) is similar to the elementary blocks of the XXZ chain
- The sum over the dynamical parameter (L terms) entirely contained inside $\bar{\mathbb{P}}(\mathbf{s},|z|-|\zeta|;\mathrm{k},\ell)$

Multi-point local height probabilities

- **Polarized basis:**
  $\bar{\mathbf{P}}(s,Z;\epsilon,\mathrm{t}) = \sum_{\mathrm{k}=0}^{1}\sum_{\ell=0}^{L-r-1}(-1)^{\mathrm{k}\epsilon}\, e^{-i\pi\frac{r\mathrm{k}+2\ell}{L-r}(\mathrm{t}+s_0)}\ \bar{\mathbb{P}}(s,Z;\mathrm{k},\ell)$
- only the analytical part is changed (resummation is possible)

$$\bar{\mathbf{P}}_{\alpha_1,\ldots,\alpha_m}(s;\epsilon,\mathrm{t}) \propto \int_{\mathcal{C}_-}\prod_{j=1}^{|\boldsymbol{\alpha}_-|}dz_j\int_{\mathcal{C}_+}\prod_{j=|\boldsymbol{\alpha}_-|+1}^{m}dz_j\ \widetilde{G}_{\alpha_1,\ldots,\alpha_m}(s;\{z\},\{\zeta\})$$

$$\times\,\bar{\mathcal{S}}_m(\{z\};\{\zeta\})\ \bar{\mathbf{P}}(s,|z|-|\zeta|;\epsilon,\mathrm{t}) + O(N^{-\infty}).$$

- MPLHP are diagonals: ground states sectors are frozen

$$\bar{\mathbf{P}}(s,Z;\epsilon,\mathrm{t})\Big|_{\substack{L\ \mathrm{even}\\ \epsilon+\mathrm{t}+s_0-s\ \mathrm{odd}}} = 0, \qquad \text{with } \tilde{s}_0 = s_0 + \frac{1}{2\tilde{\eta}} \in \mathbb{R}$$

$$\bar{\mathbf{P}}(s,Z;\epsilon,\mathrm{t})\Big|_{\substack{L\ \mathrm{even}\\ \epsilon+\mathrm{t}+s_0-s\ \mathrm{even}}} = 2e^{i\pi\left(2\frac{r}{L}\tilde{s}Z-\frac{L-r}{L}Z^2\tilde{\tau}\right)}\,\frac{\theta_4\!\left(\frac{r\tilde{s}}{L};\tau\right)\theta_3\!\left(\frac{\tilde{s}_0+\mathrm{t}}{L-r}-\frac{\tilde{s}}{2L}+\frac{Z\tau}{r};\frac{\tau}{r(L-r)}\right)}{L\,\theta_4\!\left(0;\frac{L}{r}\tau\right)\theta_4\!\left(\frac{r(\tilde{s}_0+\mathrm{t})}{L-r};\frac{L}{L-r}\tau\right)},$$

$$\bar{\mathbf{P}}(s,Z;\epsilon,\mathrm{t})\Big|_{L\ \mathrm{odd}} = e^{i\pi\left(2\frac{r}{L}\tilde{s}Z-\frac{L-r}{L}Z^2\tilde{\tau}\right)}$$

$$\times\,\frac{\theta_4\left(\frac{r\tilde{s}}{L};\tau\right)\ \theta_3\left(\left(\frac{1}{2}-\frac{1}{2L}\right)\tilde{s}-\left(\frac{1}{2}-\frac{1}{2(L-r)}\right)(\tilde{s}_0+\mathrm{t})-\frac{\epsilon}{2}+\frac{Z}{2r}\tau;\frac{\tau}{4r(L-r)}\right)}{L\,\theta_4\left(0;\frac{L}{r}\tau\right)\ \theta_4\left(\frac{r(\tilde{s}_0+\mathrm{t})}{L-r};\frac{L}{L-r}\tau\right)},$$

- single m-fold integral as for the XXZ spin chain

Conclusion and perspectives

- **Summary of the results obtained for the cyclic SOS model**
  - ⋆ Determinant representations for scalar products/norms of Bethe states/form factors in finite volume (Véronique's talk)
  - ⋆ representation as L multiple sums of determinants for the Multi-point matrix elements in finite volume
  - ⋆ Study of the thermodynamic limit:
    $\rightsquigarrow$ Explicit result for the spontaneous polarization at the thermodynamic limit
    $\rightsquigarrow$ single m-fold integral formula for the MPLHP

- **Further questions . . .**
  - ⋆ study of two-point correlation functions in the thermodynamic limit
  - ⋆ Unrestricted SOS model? ($\eta \in \mathbb{R}$)
  - ⋆ **XYZ model?**
    - $\rightsquigarrow$ combinatorial complexity of Vertex-IRF transformation