Thermal form factors of the anisotropic Heisenberg chain

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Background

- More than a decade of steady progress in our understanding of correlation functions of integrable models, chief example XXZ chain

$$H = J \sum_{j=-L+1}^{L} \Big(\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \big(\sigma_{j-1}^z \sigma_j^z - 1 \big) \Big) - \frac{h}{2} \sum_{j=-L+1}^{L} \sigma_j^z$$

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- Analysis of long distance asymptotics in the critical phase from form factor expansion on the lattice (KITANINE ET AL. 11)

Contents

- Finite temperature correlation functions for XXZ
- Form factor expansion
- Spectral problem of the quantum transfer matrix
- Form factors for longitudinal correlation functions
- Form factors for transversal correlation functions
- Derivation of form factor formulae
- Summary amd outlook

R matrix and statistical operator

- R matrix (solution of YANG-BAXTER equation)

$$R(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{aligned} b(\lambda,\mu) &= \frac{\mathrm{sh}(\lambda-\mu)}{\mathrm{sh}(\lambda-\mu+\eta)} \\ c(\lambda,\mu) &= \frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda-\mu+\eta)} \end{aligned}$$

- For the exact calculation of thermal averages we need to express the statistical operator $\mathrm{e}^{-H/T}$ in terms of $R(\lambda,\mu)$

- For this purpose we introduce an auxiliary vertex model with monodromy matrix

$$T_j(\lambda) = q^{\kappa\sigma_j^z} R_{j\overline{N}}\big(\lambda, \tfrac{\beta}{N}\big) R^{t_1}_{\overline{N-1}j}\big(-\tfrac{\beta}{N}, \lambda\big) \dots R_{j\overline{2}}\big(\lambda, \tfrac{\beta}{N}\big) R^{t_1}_{\overline{1}j}\big(-\tfrac{\beta}{N}, \lambda\big)$$

$j = -L+1, \dots, L$ and $N \in 2\mathbb{N}$. Here

$$\beta = \frac{2J\,\mathrm{sh}(\eta)}{T}, \qquad \kappa = \frac{h}{2\eta T}$$

Correlation functions

- Then

$$\mathrm{e}^{-H/T} = \lim_{N\to\infty} \underbrace{\mathrm{tr}_{\bar{1}\dots\bar{N}}\{T_{-L+1}(0)\dots T_L(0)\}}_{=\rho_{N,L}}$$

- And expectation values are approximated as

$$\langle \mathcal{O}_1^{(1)}\dots\mathcal{O}_m^{(m)}\rangle_N = \lim_{L\to\infty} \frac{\mathrm{Tr}_{-L+1\dots L}\{\rho_{N,L}\mathcal{O}_1^{(1)}\dots\mathcal{O}_m^{(m)}\}}{\mathrm{Tr}_{-L+1\dots L}\{\rho_{N,L}\}}$$

$$= \lim_{L\to\infty} \frac{\mathrm{Tr}_{\bar{1}\dots\bar{N}}\{\mathrm{Tr}^L\{T(0)\}\,\mathrm{Tr}\{\mathcal{O}^{(1)}T(0)\}\dots\mathrm{Tr}\{\mathcal{O}^{(m)}T(0)\}\,\mathrm{Tr}^{L-m}\{T(0)\}\}}{\mathrm{Tr}_{\bar{1}\dots\bar{N}}\{\mathrm{tr}^{2L}\{T(0)\}\}}$$

$$= \frac{\langle\Psi_0|\,\mathrm{Tr}\{\mathcal{O}^{(1)}T(0)\}\dots\mathrm{Tr}\{\mathcal{O}^{(m)}T(0)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0^m(0)}$$

where $\Lambda_0(0)$ is the unique eigenvalue of largest modulus of the quantum transfer matrix $t(\lambda) = \mathrm{Tr}\,T(\lambda)$ and $|\Psi_0\rangle$ is the corresponding eigenvector

Longitudinal correlations and generating function

- We shall concentrate on longitudinal and transversal two-point functions, $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ and $\langle \sigma_1^- \sigma_{m+1}^+ \rangle$

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- The longitudinal correlation functions can be obtained from a generating function $\left\langle q^{2\alpha S(m)}\right\rangle$, where $S(m) = \frac{1}{2}\sum_{j=1}^m \sigma_j^z$

$$\langle\sigma_1^z\rangle = D_m\partial_{\eta\alpha}\left\langle q^{2\alpha S(m+1)}\right\rangle\Big|_{\alpha=0}, \qquad \langle\sigma_1^z\sigma_{m+1}^z\rangle = \frac{1}{2}D_m^2\partial_{\eta\alpha}^2\left\langle q^{2\alpha S(m+1)}\right\rangle\Big|_{\alpha=0}$$

D_m is the difference operator $D_m f(m) = f(m) - f(m-1)$

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- It follows from the general formula that

$$\big\langle q^{2\alpha S(m)} \big\rangle_N = \frac{\langle \Psi_0 | t^m(0|\alpha) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \Lambda_0^m(0)} = \sum_{n=0}^{N_M - 1} A_n(\alpha) \rho_n^m(0|\alpha)$$

  Here $t^m(0|\alpha)|\Psi_0\rangle$ was expanded in the basis of Bethe states, and

$$A_n(\alpha) = \frac{\langle \Psi_0 | \Psi_n^\alpha \rangle \langle \Psi_n^\alpha | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \langle \Psi_n^\alpha | \Psi_n^\alpha \rangle}, \quad \rho_n(\lambda|\alpha) = \frac{\Lambda_n(\lambda|\alpha)}{\Lambda_0(\lambda)}$$

  $t(\lambda|\alpha)$ is obtained from $t(\lambda)$ by shifting $\kappa \to \kappa + \alpha$

Longitudinal correlations and generating function

- Taking derivatives we obtain

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_N - \langle \sigma_1^z \rangle_N \langle \sigma_{m+1}^z \rangle_N = \sum_{n=1}^{N_M - 1} A_n^{zz} \left(\rho_n^{\frac{1}{2}} - \rho_n^{-\frac{1}{2}} \right)^2 \rho_n^m$$

In this expression we used the abbreviations

$$\rho_n = \rho_n(0|0) \, , \quad A_n^{zz} = \frac{1}{2} \partial_{\eta\alpha}^2 A_n(\alpha) \big|_{\alpha=0} = \frac{\langle \Psi_0 | \Psi'_n \rangle \langle \Psi'_n | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \langle \Psi_n | \Psi_n \rangle} \, .$$

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- $1 > |\rho_1| \geq |\rho_2| \geq \ldots$ by construction. Expansion is the finite temperature asymptotic expansion of the longitudinal correlation functions. The leading asymptotics is determined by the first few terms. The correlation length of the longitudinal correlation functions is

$$\xi_1 = -\frac{1}{\ln|\rho_1|}\,. \qquad (1)$$

  The correlation length was studied in (TAKAHASHI 91, KLÜMPER 93 etc.). So far the amplitudes in were only studied numerically (FABRICIUS, KLÜMPER and MCCOY 99) for finite Trotter numbers

Transversal correlations

- Similarly we obtain a finite temperature asymptotic expansion for the transversal correlation functions

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle_N = \sum_{n=1}^{N_M} A_n^{-+} \rho_n^m ,$$

where

$$A_n^{-+} = \frac{\langle \Psi_0 | B(0) | \Psi_n \rangle}{\Lambda_n(0) \langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n | C(0) | \Psi_0 \rangle}{\Lambda_0(0) \langle \Psi_n | \Psi_n \rangle}$$

- Bottom line: Thermal correlation functions can be expanded into series of form factors of the quantum transfer matrix. Instaed of form factors of local operators, form factors of ABA operators appear.

Algebraic Bethe ansatz solution

- The monodromy matrix $T(\lambda)$ is a 2×2 matrix with matrix elements $A(\lambda), B(\lambda), C(\lambda), D(\lambda) \in \mathrm{End}\big(\mathbb{C}_2^{\otimes N}\big)$. The quantum transfer matrix $\mathrm{Tr}\, T(\lambda) = A(\lambda) + D(\lambda)$ can be diagonalized by the algebraic Bethe ansatz:

$$|\Psi_n\rangle = B(\lambda_M) \dots B(\lambda_1)|0\rangle$$

is an eigenvector of the quantum transfer matrix if the Bethe roots λ_j, $j = 1, \dots, M$, satisfy the system

$$\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\mathrm{sh}(\lambda_j - \lambda_k + \eta)}{\mathrm{sh}(\lambda_j - \lambda_k - \eta)}$$

of Bethe ansatz equations. The corresponding eigenvalue is

$$\Lambda_n(\lambda) = a(\lambda) \prod_{j=1}^{M} \frac{\mathrm{sh}(\lambda - \lambda_j - \eta)}{\mathrm{sh}(\lambda - \lambda_j)} + d(\lambda) \prod_{j=1}^{M} \frac{\mathrm{sh}(\lambda - \lambda_j + \eta)}{\mathrm{sh}(\lambda - \lambda_j)}$$

A left eigenvector of the quantum transfer matrix with the same eigenvalue is

$$\langle \Psi_n| = \langle 0|C(\lambda_1) \dots C(\lambda_M)$$

Nonlinear integral equations

- For every solution of the Bethe equations we define an auxiliary function

$$\mathfrak{a}_n(\lambda) = \frac{d(\lambda)}{a(\lambda)} \prod_{j=1}^{M} \frac{\operatorname{sh}(\lambda - \lambda_j + \eta)}{\operatorname{sh}(\lambda - \lambda_j - \eta)}$$

  In the twisted case we write $\mathfrak{a}_n(\lambda|\alpha)$.

- In the Trotter limit the functions $\mathfrak{a}_n(\lambda)$ are uniquely determined by the integral equation (KLÜMPER 93)

$$\ln(\mathfrak{a}_n(\lambda)) = -(2\kappa + N - 2M)\eta - \beta \mathrm{e}(\lambda) - \int_{\mathcal{C}_n} \frac{\mathrm{d}\mu}{2\pi \mathrm{i}} K(\lambda - \mu) \ln\bigl(1 + \mathfrak{a}_n(\mu)\bigr)$$

  Here we have introduced the kernel

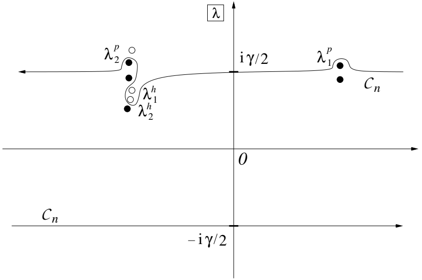
$$K(\lambda) = K_0(\lambda)\,, \quad K_\alpha(\lambda) = q^{-\alpha} \operatorname{cth}(\lambda - \eta) - q^{\alpha} \operatorname{cth}(\lambda + \eta)$$

  which is the derivative of the bare two-particle scattering phase, and the bare energy

$$\mathrm{e}(\lambda) = \operatorname{cth}(\lambda) - \operatorname{cth}(\lambda + \eta)$$

Contour and functions ρ_n

- The Contour $\mathcal{C}_n$ (here $\eta = -i\gamma$)



- The functions $\rho_n(\lambda|\alpha)$ can be represented as integrals over the auxiliary functions

$$\rho_n(\lambda|\alpha) = q^{\alpha + \frac{N}{2} - M} \exp\left\{ \int_{\mathcal{C}_n} \frac{d\mu}{2\pi i} e(\mu - \lambda) \ln\left(\frac{1 + \mathfrak{a}_n(\mu|\alpha)}{1 + \mathfrak{a}_0(\mu)} \right) \right\}$$

where λ is located inside the contour $\mathcal{C}_n$. The number $N/2 - M$ is the eigenvalue of the conserved z-component of the pseudo spin $\eta^z = \frac{1}{2} \sum_{j=1}^{N} (-1)^j \sigma_j^z$

Amplitudes of the generating function

- Parameterized by Bethe roots

$$A_n(\alpha) = \left[\prod_{j=1}^{M} \frac{\rho_n(\lambda_j|\alpha)}{\rho_n(\mu_j|\alpha)}\right]$$

$$\times \frac{\det_M\left\{\delta^j_k - \frac{\rho_n(\mu_j|\alpha)}{\mathfrak{a}'_n(\mu_j|\alpha)}\mathcal{K}_{-\alpha}(\mu_j - \mu_k)\right\}}{\det_M\left\{\delta^j_k - \frac{1}{\mathfrak{a}'_n(\mu_j|\alpha)}\mathcal{K}(\mu_j - \mu_k)\right\}} \frac{\det_M\left\{\delta^j_k - \frac{\rho_n^{-1}(\lambda_j|\alpha)}{\mathfrak{a}'_0(\lambda_j)}\mathcal{K}_\alpha(\lambda_j - \lambda_k)\right\}}{\det_M\left\{\delta^j_k - \frac{1}{\mathfrak{a}'_0(\lambda_j)}\mathcal{K}(\lambda_j - \lambda_k)\right\}}$$

Here $M = N/2$, the λ_j, $j = 1, \dots, M$, are the Bethe roots of the dominant state, and the μ_j, $j = 1, \dots, M$, are the Bethe roots of an excited state of the twisted transfer matrix $t(\lambda|\alpha)$. The kernel functions are defined as

$$\mathcal{K}(\lambda) = \mathcal{K}_0(\lambda), \quad \mathcal{K}_\alpha(\lambda) = \frac{e^{(\alpha-1)(\lambda-\eta)}}{\mathrm{sh}(\lambda-\eta)} - \frac{e^{(\alpha-1)(\lambda+\eta)}}{\mathrm{sh}(\lambda+\eta)}$$

(same as in BJMS 10)

Trotter limit

- Prefactor

$$\prod_{j=1}^{M} \frac{\rho_n(\lambda_j|\alpha)}{\rho_n(\mu_j|\alpha)} = \exp\left\{ - \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i} \ln\big(\rho_n(\lambda|\alpha)\big) \partial_\lambda \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\}$$

- The determinants are all of the same structure. Expanding e.g. the first determinant in the numerator we obtain

$$\det_M \left\{ \delta^j_k - \frac{\rho_n(\mu_j|\alpha)}{\mathfrak{a}'_n(\mu_j|\alpha)} \mathcal{K}_{-\alpha}(\mu_j - \mu_k) \right\} = 1 - \sum_{j=1}^{M} \frac{\rho_n(\mu_j|\alpha)}{\mathfrak{a}'_n(\mu_j|\alpha)} \mathcal{K}_{-\alpha}(0)$$

$$+ \sum_{1 \le j < k \le M} \frac{\rho_n(\mu_j|\alpha)}{\mathfrak{a}'_n(\mu_j|\alpha)} \frac{\rho_n(\mu_k|\alpha)}{\mathfrak{a}'_n(\mu_k|\alpha)} \det \begin{vmatrix} \mathcal{K}_{-\alpha}(0) & \mathcal{K}_{-\alpha}(\mu_j - \mu_k) \\ \mathcal{K}_{-\alpha}(\mu_k - \mu_j) & \mathcal{K}_{-\alpha}(0) \end{vmatrix} - \dots$$

$$= 1 + \sum_{k=1}^{M} \frac{(-1)^k}{k!} \left[\prod_{j=1}^{k} \int_{\mathcal{C}_n} dm_+^\alpha(\nu_j) \right] \det_k \{ \mathcal{K}_{-\alpha}(\nu_l - \nu_m) \}$$

where we have introduced the 'measure'

$$dm_+^\alpha(\lambda) = \frac{d\lambda\, \rho_n(\lambda|\alpha)}{2\pi i (1 + \mathfrak{a}_n(\lambda|\alpha))}$$

Trotter limit

- In the Trotter limit $N = 2M \to \infty$ this converges to the Fredholm determinant of the integral operator $\widehat{\mathcal{K}}_{-\alpha}$ defined by the kernel $\mathcal{K}_{-\alpha}$, the measure $\mathrm{d}m_+^\alpha$ and the contour $\mathcal{C}_n$,

$$\lim_{N\to\infty} \det_M \left\{ \delta^j_k - \frac{\rho_n(\mu_j|\alpha)}{\mathfrak{a}_n'(\mu_j|\alpha)} \mathcal{K}_{-\alpha}(\mu_j - \mu_k) \right\}$$
$$= 1 + \sum_{k=1}^\infty \frac{(-1)^k}{k!} \left[\prod_{j=1}^k \int_{\mathcal{C}_n} \mathrm{d}m_+^\alpha(\nu_j) \right] \det_k \{ \mathcal{K}_{-\alpha}(\nu_l - \nu_m) \}$$
$$= \det_{\mathrm{d}m_+^\alpha, \mathcal{C}_n} \{ 1 - \widehat{\mathcal{K}}_{-\alpha} \}$$

The other determinants can be treated in a similar way. We introduce the measures

$$\mathrm{d}m_-^\alpha(\lambda) = \frac{\mathrm{d}\lambda\, \rho_n^{-1}(\lambda|\alpha)}{2\pi\mathrm{i}(1 + \mathfrak{a}_0(\lambda))}$$
$$\mathrm{d}m(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}(1 + \mathfrak{a}_0(\lambda))}, \quad \mathrm{d}m_0^\alpha(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}(1 + \mathfrak{a}_n(\lambda|\alpha))}$$

Trotter limit

- Then

$$A_n(\alpha) = \exp\left\{ - \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i} \ln\bigl(\rho_n(\lambda|\alpha)\bigr) \partial_\lambda \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\}$$
$$\times \frac{\det_{dm_+^\alpha, \mathcal{C}_n}\{1 - \widehat{\mathcal{K}}_{-\alpha}\} \det_{dm_-^\alpha, \mathcal{C}_n}\{1 - \widehat{\mathcal{K}}_\alpha\}}{\det_{dm_0^\alpha, \mathcal{C}_n}\{1 - \widehat{\mathcal{K}}\} \det_{dm, \mathcal{C}_n}\{1 - \widehat{\mathcal{K}}\}} .$$

  for the amplitudes in the Trotter limit.

- The amplitudes are entirely described in terms of functions which appeared earlier in the description of the thermodynamic properties, the correlation lengths and the correlation functions of the model. These are the auxiliary functions $\mathfrak{a}_n$ (KLÜMPER 93), the eigenvalue ratios ρ_n (BOOS, FG 09, JIMBO, MIWA and SMIRNOV 09) and the deformed kernel $\mathcal{K}_\alpha$ (BOOS ET AL. 07).

Transversal case

- In the transversal case we consider

$$A_n^{-+}(\xi) = \frac{\langle \Psi_0 | B(\xi) | \Psi_n^\alpha \rangle}{\Lambda_n(\xi|\alpha) \langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n^\alpha | C(\xi) | \Psi_0 \rangle}{\Lambda_0(\xi) \langle \Psi_n^\alpha | \Psi_n^\alpha \rangle}$$

- We obtain the following formula in the Trotter limit

$$A_n^{-+}(\xi) = \frac{\overline{G}_+^-(\xi)\overline{G}_-^+(\xi)}{(q^{1+\alpha} - q^{-1-\alpha})(q^\alpha - q^{-\alpha})}$$

$$\times \exp\left\{ - \int_{\mathcal{C}_n} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} \ln\big(\rho_n(\lambda|\alpha)\big) \partial_\lambda \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\}$$

$$\times \frac{\det_{\mathrm{d}m_+^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}_{1-\alpha}\right\} \det_{\mathrm{d}m_-^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}_{1+\alpha}\right\}}{\det_{\mathrm{d}m_0^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}\right\} \det_{\mathrm{d}m, \mathcal{C}_n}\left\{1 - \widehat{K}\right\}}$$

The G functions

- Here, for $s = \pm$,

$$\overline{G}_s^{\pm}(\xi) = \lim_{\mathrm{Re}\,\lambda \to \pm\infty} \overline{G}_s^{\pm}(\lambda, \xi)$$

and $\overline{G}_s(\lambda, \xi)$ is the solution of the linear integral equation

$$\overline{G}_s(\lambda, \xi) = -\operatorname{cth}(\lambda - \xi) + q^{\alpha - s} \rho_n^s(\xi|\alpha) \operatorname{cth}(\lambda - \xi - \eta) + \int_{\mathcal{C}_n} \mathrm{d}m_s^{\alpha}(\mu) \overline{G}_s(\mu, \xi) K_{\alpha - s}(\mu - \lambda)$$

Longitudinal case

- The derivation of the form factor formulae is based on the scalar product formula (SLAVNOV 89)

$$\langle 0|C(\mu_1)\dots C(\mu_M)B(\lambda_M)\dots B(\lambda_1)|0\rangle = \langle 0|C(\lambda_1)\dots C(\lambda_M)B(\mu_M)\dots B(\mu_1)|0\rangle$$

$$= \left[\prod_{j=1}^M a(\mu_j)d(\lambda_j)\prod_{k=1}^M \frac{1}{b(\lambda_j,\mu_k)}\right] \frac{\det_M\big(\mathrm{e}(\lambda_j-\mu_k)-\mathrm{e}(\mu_k-\lambda_j)\mathfrak{a}_n(\mu_k)\big)}{\det_M\Big(\frac{1}{\mathrm{sh}(\lambda_j-\mu_k)}\Big)}$$

Here the λ_j, $j=1,\dots,M$, are a solution of the Bethe equations and $\mathfrak{a}_n$ is the associated auxiliary function. The μ_j in this formula are free. They may or may not be a solution of the Bethe equations. In particular, we can take the limit $\mu_j \to \lambda_j$ and obtain the 'norm formula' for Bethe states

$$\langle \Psi_n|\Psi_n\rangle = \left[\prod_{j=1}^M d(\lambda_j)\Lambda_n(\lambda_j)\right] \det_M\left\{\delta^j_k - \frac{K(\lambda_j-\lambda_k)}{\mathfrak{a}'_n(\lambda_j)}\right\}$$

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- Task: take Trotter limit $N\to\infty$. Norm formula good, scalar product formula not yet

Longitudinal case

- Scalar products between two different Bethe states are generally vanishing. But if we consider e.g. the dominant state $|\Psi_0\rangle$ wth Bethe roots $\{\lambda_j\}$ and a twisted Bethe state $|\Psi_n^\alpha\rangle$ with Bethe roots $\{\mu_k\}$ we can rewrite the scalar product as

$$\langle \Psi_n^\alpha | \Psi_0 \rangle = \left[\prod_{j=1}^{M} q^\alpha d(\lambda_j) \Lambda_0(\mu_j) e^{\mu_j - \lambda_j} \right] \det_M \left\{ \delta_k^j - \frac{\mathcal{K}_{-\alpha}(\mu_j - \mu_k)}{\rho_n^{-1}(\mu_j|\alpha) \mathfrak{a}_n'(\mu_j|\alpha)} \right\}$$

Longitudinal case

- Scalar products between two different Bethe states are generally vanishing. But if we consider e.g. the dominant state $|\Psi_0\rangle$ wth Bethe roots $\{\lambda_j\}$ and a twisted Bethe state $|\Psi_n^\alpha\rangle$ with Bethe roots $\{\mu_k\}$ we can rewrite the scalar product as

$$\langle\Psi_n^\alpha|\Psi_0\rangle = \left[\prod_{j=1}^M q^\alpha d(\lambda_j)\Lambda_0(\mu_j)e^{\mu_j-\lambda_j}\right]\det_M\left\{\delta_k^j - \frac{\mathcal{K}_{-\alpha}(\mu_j-\mu_k)}{\rho_n^{-1}(\mu_j|\alpha)\mathfrak{a}_n'(\mu_j|\alpha)}\right\}$$

- From here we recover the norm formula by setting $\lambda_j = \mu_j$ and sending α to zero. Interchanging λs and μs we obtain the four factors occuring in the amplitudes

$$A_n(\alpha) = \frac{\langle\Psi_0|\Psi_n^\alpha\rangle\langle\Psi_n^\alpha|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\langle\Psi_n^\alpha|\Psi_n^\alpha\rangle}$$

Longitudinal case

- For the derivation we introduce the ratio of Q-functions

$$\phi(\lambda) = \prod_{j=1}^{M} \frac{\operatorname{sh}(\lambda - \mu_j)}{\operatorname{sh}(\lambda - \lambda_j)} .$$

- It satisfies the two identities

$$\phi(\mu_k - \eta)\phi^{-1}(\mu_k + \eta) = -q^{-2\alpha}\mathfrak{a}_0(\mu_k)$$

$$\phi'(\mu_j)\big(q^{-\alpha}\phi^{-1}(\mu_j - \eta) - q^{\alpha}\phi^{-1}(\mu_j + \eta)\big) = \rho_n^{-1}(\mu_j|\alpha)\mathfrak{a}_n'(\mu_j|\alpha)$$

which follow from the Bethe ansatz equations for the μ_k and from Baxter's *TQ*-equation

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which follow from the Bethe ansatz equations for the μ_k and from Baxter's *TQ*-equation

- In terms of ϕ

$$\left[\prod_{j=1}^{M} \frac{d(\mu_j)}{d(\lambda_j)}\right] \frac{\langle \Psi_n^\alpha | \Psi_0 \rangle}{\langle \Psi_n^\alpha | \Psi_n^\alpha \rangle} = \left[\prod_{j=1}^{M} \phi^{-1}(\mu_k - \eta)\right]$$

$$\times \det_M \left\{ \frac{\mathrm{res}\,\phi(\lambda_k)}{\mathrm{sh}(\mu_j - \lambda_k)} \right\} \frac{\det_M \{ \mathrm{e}(\lambda_j - \mu_k) - \mathrm{e}(\mu_k - \lambda_j)\mathfrak{a}_0(\mu_k) \}}{\det_M \{ \delta_k^j \mathfrak{a}_n'(\mu_j|\alpha) - K(\mu_j - \mu_k) \}}$$

Longitudinal case

Now the result follows by means of the norm formula and a trick which appeared in similar form in KITANINE, MAILLET AND TERRAS 99:

$$\left[\prod_{j=1}^{M} q^{-\alpha}\phi^{-1}(\mu_k-\eta)\right]\det_M\left\{\frac{e^{\lambda_\ell-\mu_j}\operatorname{res}\phi(\lambda_\ell)}{\operatorname{sh}(\mu_j-\lambda_\ell)}\right\}\det_M\{\mathrm{e}(\lambda_j-\mu_k)-\mathrm{e}(\mu_k-\lambda_j)\mathfrak{a}_0(\mu_k)\}$$

$$=\det_M\left\{\frac{e^{\lambda_\ell-\mu_j}\operatorname{res}\phi(\lambda_\ell)}{\operatorname{sh}(\mu_j-\lambda_\ell)}\right\}\det_M\{q^{-\alpha}\phi^{-1}(\mu_k-\eta)\mathrm{e}(\lambda_\ell-\mu_k)+q^{\alpha}\phi^{-1}(\mu_k+\eta)\mathrm{e}(\mu_k-\lambda_\ell)\}$$

$$=\det_M\left\{\int_{\mathcal{C}}\frac{d\lambda}{2\pi i}\,\frac{e^{\lambda-\mu_j}\phi(\lambda)}{\operatorname{sh}(\mu_j-\lambda)}\left(q^{-\alpha}\phi^{-1}(\mu_k-\eta)\mathrm{e}(\lambda-\mu_k)+q^{\alpha}\phi^{-1}(\mu_k+\eta)\mathrm{e}(\mu_k-\lambda)\right)\right\}$$

$$=\det_M\left\{\delta_k^j\rho_n^{-1}(\mu_j|\alpha)\mathfrak{a}_n'(\mu_j|\alpha)-\frac{q^{1+\alpha}}{\operatorname{sh}(\mu_j-\mu_k-\eta)}+\frac{q^{-1-\alpha}}{\operatorname{sh}(\mu_j-\mu_k+\eta)}\right\}$$

The contour $\mathcal{C}$ in the second equation is chosen in such a way that all Bethe roots λ_j, $j=1,\dots,M$, are included, but μ_j,μ_k and $\mu_k\pm\eta$ are excluded. The integral can be calculated, since the integrand is an $i\pi$-periodic function of λ which decreases to zero as $\operatorname{Re}\lambda\to\pm\infty$

Remarks on the transversal case

- Derivation more involved in the transversal case

$$A_n^{-+}(\xi) = F_-(\xi) F_+(\xi)$$

where

$$F_+(\xi) = \frac{\langle \Psi_n^\alpha | C(\xi) | \Psi_0 \rangle}{\Lambda_0(\xi) \langle \Psi_n^\alpha | \Psi_n^\alpha \rangle} = \frac{\langle 0 | C(\mu_1) \dots C(\mu_{M-1}) C(\xi) B(\lambda_M) \dots B(\lambda_1) | 0 \rangle}{\Lambda_0(\xi) \langle 0 | C(\mu_1) \dots C(\mu_{M-1}) B(\mu_{M-1}) \dots B(\mu_1) | 0 \rangle}$$

$$F_-(\xi) = \frac{\langle \Psi_0 | B(\xi) | \Psi_n^\alpha \rangle}{\Lambda_n(\xi) \langle \Psi_0 | \Psi_0 \rangle} = \frac{\langle 0 | C(\lambda_1) \dots C(\lambda_M) B(\xi) B(\mu_{M-1}) \dots B(\mu_1) | 0 \rangle}{\Lambda_n(\xi) \langle 0 | C(\lambda_1) \dots C(\lambda_M) B(\lambda_M) \dots B(\lambda_1) | 0 \rangle}$$

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- Clearly we can use the scalar product formula for $|\Psi_0\rangle$ being a Bethe vector. It implies, in particular,

$$F_-(\xi) = \frac{\langle \Psi_n^\alpha | \Psi_n^\alpha \rangle F_+(\xi)}{\rho_n(\xi|\alpha) \langle \Psi_0 | \Psi_0 \rangle}$$

Hence, enough to calculate $F_+(\xi)$. Then, however, the result for the amplitude is not of symmetric form

Remarks on the transversal case

- If we want to use the scalar product formula for $|\Psi_n^\alpha\rangle$ being a Bethe vector in

$$F_-(\xi) = \frac{\langle\Psi_0|B(\xi)|\Psi_n^\alpha\rangle}{\Lambda_n(\xi)\langle\Psi_0|\Psi_0\rangle} = \frac{\langle 0|C(\lambda_1)\dots C(\lambda_M)B(\xi)B(\mu_{M-1})\dots B(\mu_1)|0\rangle}{\Lambda_n(\xi)\langle 0|C(\lambda_1)\dots C(\lambda_M)B(\lambda_M)\dots B(\lambda_1)|0\rangle}$$

  we have to move $B(\xi)$ through the product of C operators in $\langle\Psi_0|$ to the left using the Yang-Baxter algebra relations. This produces a double sum

$$F_-(\xi) = \sum_{\substack{\ell,m \\ \ell\neq m}}^{M+1} d(\lambda_\ell)a(\lambda_m)c(\lambda_\ell,\xi)c(\xi,\lambda_m) \\ \times \left[\prod_{\substack{k=1 \\ k\neq\ell}}^{M+1} \frac{1}{b(\lambda_\ell,\lambda_k)}\right]\left[\prod_{\substack{k=1 \\ k\neq\ell,m}}^{M+1} \frac{1}{b(\lambda_k,\lambda_m)}\right] \frac{\langle 0|\prod_{\substack{k=1 \\ k\neq\ell,m}}^{M+1} C(\lambda_k)|\Psi_n^\alpha\rangle}{\Lambda_n(\xi|\alpha)\langle\Psi_0|\Psi_0\rangle}$$

  where $\lambda_{M+1} = \xi$

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- If we want to use the scalar product formula for $|\Psi_n^\alpha\rangle$ being a Bethe vector in

$$F_-(\xi) = \frac{\langle\Psi_0|B(\xi)|\Psi_n^\alpha\rangle}{\Lambda_n(\xi)\langle\Psi_0|\Psi_0\rangle} = \frac{\langle 0|C(\lambda_1)\dots C(\lambda_M)B(\xi)B(\mu_{M-1})\dots B(\mu_1)|0\rangle}{\Lambda_n(\xi)\langle 0|C(\lambda_1)\dots C(\lambda_M)B(\lambda_M)\dots B(\lambda_1)|0\rangle}$$

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  where $\lambda_{M+1} = \xi$

- Here we can use the the scalar product formula in the last term under the sum

Remarks on the transversal case

- After a slightly cumbersome calculation

$$F_-(\xi) = \frac{e^{-\xi} \prod_{j=1}^{M-1} q^{-\alpha} d(\mu_j) e^{\mu_j}}{\prod_{j=1}^{M} d(\lambda_j) e^{\lambda_j} \rho_n^{-1}(\lambda_j|\alpha)} \frac{\det_M \left\{ \delta_k^j - \frac{\rho_n^{-1}(\lambda_j|\alpha)}{\mathfrak{a}_0'(\lambda_j|\alpha)} K_{1+\alpha}(\lambda_j - \lambda_k) \right\}}{\det_M \left\{ \delta_k^j - \frac{1}{\mathfrak{a}_0'(\lambda_j|\alpha)} K(\lambda_j - \lambda_k) \right\}}$$
$$\times \int_{\mathcal{C}_n} \mathrm{d}m_-^\alpha(\lambda) e^{\lambda} \int_{\mathcal{C}_n} \mathrm{d}\overline{m}_-^\alpha(\mu) e^{\mu} \frac{G_-(\lambda,\xi)\sigma(\mu) - G_-(\mu,\xi)\sigma(\lambda)}{\mathrm{sh}(\lambda - \mu - \eta)}$$

where

$$G_-(\lambda,\xi) = -\mathrm{cth}(\lambda - \xi) + q^{-\alpha-1} \rho_n^{-1}(\xi|\alpha) \mathrm{cth}(\lambda - \xi - \eta) + \int_{\mathcal{C}_n} \mathrm{d}m_-^\alpha(\mu) K_{1+\alpha}(\lambda - \mu) G_-(\mu,\xi)$$

$$\sigma(\lambda) = 1 + \int_{\mathcal{C}_n} \mathrm{d}m_-^\alpha(\mu) K_{1+\alpha}(\lambda - \mu) \sigma(\mu)$$

- Prefactor cancels unwanted part of prefactor in $F_+(\xi)$
- Double integral stems from double sum on previous slide

Remarks on the transversal case

- Factorization occurs

$$\int_{\mathcal{C}_n} \mathrm{d}m_-^\alpha(\lambda) e^\lambda \int_{\mathcal{C}_n} \mathrm{d}\overline{m}_-^\alpha(\mu) e^\mu \, \frac{G_-(\lambda,\xi)\sigma(\mu) - G_-(\mu,\xi)\sigma(\lambda)}{\mathrm{sh}(\lambda-\mu-\eta)} = \frac{e^{2\xi}\overline{G}_-^+(\xi)}{(q^\alpha - q^{-\alpha})(q^{\alpha+1} - q^{-\alpha-1})}$$

where

$$\overline{G}_-(\lambda,\xi) = -\mathrm{cth}(\lambda-\xi) + q^{\alpha+1}\rho_n^{-1}(\xi|\alpha)\,\mathrm{cth}(\lambda-\xi-\eta) + \int_{\mathcal{C}_n} \mathrm{d}m_-^\alpha(\mu)\,\overline{G}_-(\mu,\xi) K_{\alpha+1}(\mu-\lambda)$$

Summary and Outlook

- We have derived expressions for thermal form factors of the XXZ chain in the Trotter limit
- These determine the amplitude in the asymptotic expansion of the correlation functions at finite temperature
- The formulae for the amplitudes seem to have a universal form which inspires speculations about the matrix elements of composed local operators (like $\langle \Psi_0 | B(\xi_1) B(\xi_2) | \Psi_n^\alpha \rangle$ corresponding to $\sigma_1^- \sigma_2^-$)
- We can extract the leading high-temperature behaviour and the critical behaviour at $T \to 0$ from our formulae
- The results have potential applications in the calculation of the static structure factor of the model and in the study of temperature driven cross-over phenomena

Based on joint work (JSTAT 2013 P07010) with MAXIME DUGAVE (Wuppertal) and KAROL K. KOZLOWSKI (Dijon)