On one-point functions for quantum sinh-Gordon model.

Fedor Smirnov

Joint work with Stefano Negro
1. Integral representation of the one-point function

My final goal is to compute the following infinite-fold integral:

\[ \langle \Phi_\alpha(0) \rangle_R \]

\[ = \int \prod_{j=-\infty}^{\infty} d\theta_j \prod_{j=-\infty}^{\infty} Q^2(\theta_j) e^{(\tilde{\nu} + \nu)\alpha \theta_j} \prod_{i < j} \sinh \nu (\theta_i - \theta_j) \sinh \tilde{\nu} (\theta_i - \theta_j), \]

where

\[ \frac{1}{\nu} + \frac{1}{\tilde{\nu}} = 1, \quad \nu = 1 + b^2, \quad \tilde{\nu} = 1 + b^{-2}, \]

the function \( Q(\theta) \) is defined by three requirements.
Quantum Wronskian equation

\[ Q(\theta + \frac{\pi i}{2})Q(\theta - \frac{\pi i}{2}) - Q(\theta + \frac{\pi i(\nu-2)}{2\nu})Q(\theta - \frac{\pi i(\nu-2)}{2\nu}) = 1. \]

Asymptotics

\[ \log Q(\theta) = -\rho \cosh \theta + O(1), \quad \theta \to \pm \infty. \]

Ground state: no zeros in the strip \(|\text{Im}(\theta)| \leq \pi\).

For the final formula to be applicable only the first requirement really count, two others may be eased considerably.
2. Meaning of the integral, semiclassical consideration

Consider the sinh-Gordon model with the action:

\[ A = \int \left\{ \frac{1}{4\pi} \partial_z \varphi(z, \bar{z}) \partial_{\bar{z}} \varphi(z, \bar{z}) + \frac{2\mu^2}{\sin \pi b^2} \cosh(b\varphi(z, \bar{z})) \right\} \frac{idz \wedge d\bar{z}}{2}, \]

on the cylinder

\[ C = \mathbb{C}/2\pi i R \mathbb{Z}. \]

Then the main integral gives the SoV representation of the functional integral

\[ \langle \Phi_\alpha(0) \rangle_R = \int e^{-A + a\varphi(0)} \prod_{z, \bar{z} \in C} \mathcal{D}\varphi(z, \bar{z}), \]

with the convention

\[ a = \frac{1}{2} (b + b^{-1}) \alpha. \]
For the sake of classical limit $\hbar = b^2 \to 0$ it is convenient to introduce $\phi(z, \bar{z}) = b \varphi(z, \bar{z})$. Then the classical action is

$$A = \frac{1}{4\pi} \int \left\{ \left[ \partial_z \phi(z, \bar{z}) \partial_{\bar{z}} \phi(z, \bar{z}) + 2m^2 \cosh(\phi(z, \bar{z})) \right] \frac{idz \wedge d\bar{z}}{2} \right\},$$

where $m^2 = \mu^2 \frac{16\pi b^2}{\sin \pi b^2}$ is semi-classically finite.

The main contribution to the functional integral is given by regularised action evaluated on the classical solution

$$\partial_z \partial_{\bar{z}} \phi^{cl}(z, \bar{z}) = \frac{m}{2} \sinh \phi^{cl}(z, \bar{z}),$$

rapidly decreasing at infinity, and possessing the singularity at 0:

$$\phi^{cl}(z, \bar{z}) \simeq 2\alpha \log |z|, \quad z \to 0.$$
This can be computed (Lukyanov):

\[
\mathcal{A}^{\text{cl}} = \frac{\alpha^2}{2} \log \left( \frac{m}{4} \right) + \int_0^\infty \frac{dt}{t} \left( \frac{\sinh^2(\alpha t)}{t \sinh(2t)} - \frac{\alpha^2}{2} e^{-2t} \right)
\]

\[
- \int_0^\alpha d\alpha \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \log \left( \frac{1 - e^{-r \cosh \theta - \pi i \alpha}}{1 - e^{-r \cosh \theta + \pi i \alpha}} \right),
\]

where \( r = 2\pi mR \). With this we have:

\[
\langle \Phi_{\alpha}(0) \rangle_R \sim e^{-\frac{1}{b^2} \mathcal{A}^{\text{cl}}}. \]
3. Hamiltonian approach

Consider the Hamiltonian picture with time going along the cylinder, and space along $\Gamma$ (Matsubara). Then

$$\langle \Phi_\alpha(0) \rangle_R = \langle \Psi | \Phi_\alpha(0) | \Psi \rangle,$$

where $\Psi$ is the ground state corresponding to the maximal eigenvalue of the Matsubara transfer-matrix.
4. Classics

Sinh-Gordon equation is equivalent to the zero-curvature condition for the connection:

\[
L = \partial_z + \frac{1}{4} \partial_z \phi \sigma^3 - \lambda \frac{m}{2} \left( e^{\phi} \sigma^+ + e^{-\phi} \sigma^- \right),
\]

\[
\overline{L} = \partial_{\overline{z}} - \frac{1}{4} \partial_{\overline{z}} \phi \sigma^3 - \frac{1}{\lambda} \frac{m}{2} \left( e^{\phi} \sigma^- + e^{-\phi} \sigma^+ \right).
\]

We construct the monodromy matrix:

\[
M(\lambda, z) = P \exp \int_{z}^{z+2\pi i R} (Ldz + \overline{L}d\overline{z}) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.
\]

Its trace \( T(\lambda) = \text{Tr} M(\lambda, z) \), does not depend on \( z \), this is the generating function of integrals of motion. The matrix elements of \( M(\lambda, z) \) are single-valued functions on the hyper-elliptic curve of infinite genus:

\[
T(\lambda) = \mu + \frac{1}{\mu}.
\]
Separated variables. Zeros of the function $B(\lambda)$ are real. We order them and denote by $\lambda_j, -\infty < j < \infty$. These zeros depend on $y$. They either oscillate inside the zones $|T(\lambda)| > 2$ or stay at double points $|T(\lambda)| = 2$. The variables $\log \lambda_j$ and $\log \mu_j$ are canonical:

$$pdq = \sum_{j=-\infty}^{\infty} \log \mu_j d \log \lambda_j.$$ 

Consider zeros of $T(\lambda)$ ($\tau_j$) as one half of coordinates on the phase space. Then we have for Liouville measure

$$(dp \wedge dq)_{\frac{\infty}{2}} = \prod_{j=-\infty}^{\infty} \frac{1}{\mu_j - \mu_{j-1}} \prod_{i<j}(\lambda_i^2 - \lambda_j^2) \bigwedge_{j=-\infty}^{\infty} d \log \lambda_j \bigwedge_{j=-\infty}^{\infty} d \tau_j.$$

It is convenient to switch to the variables $\theta_j = \log \lambda_j$.

Now everything is prepared for writing the semi-classical expression for the matrix elements of an operator $O(\{\theta_j\})$. There is a cohomological argument for considering only this kind of operators.
We have
\[
\langle \Psi_1 | \mathcal{O} | \Psi_2 \rangle = \int \prod_{j=-\infty}^{\infty} d\theta_j \prod_{j=-\infty}^{\infty} Q_1(\theta_j)Q_2(\theta_j)e^{\frac{1}{b^2}2j\theta_j} \prod_{i<j} \sinh(\theta_i - \theta_j),
\]
where
\[
Q(\theta_j) \simeq \frac{1}{(\mu_j - \mu_j^{-1})^\frac{1}{2}} \exp\left\{ \frac{1}{ib^2} \int \log \mu \, d\log \lambda \right\}.
\]
We consider the simplest case when both $\Psi_{1,2}$ are the ground state. For this case
\[
\mu = e^{\frac{1}{2} mR(\lambda - \lambda^{-1})},
\]
which implies
\[
Q(\theta) \simeq \frac{1}{(\sinh(mR \sinh \theta))^{\frac{1}{2}}} e^{-\frac{1}{b^2} mR \cosh \theta}.
\]
For the operator \( \mathcal{O} \) we take

\[
\Phi_\alpha(0) = e^{\frac{1}{b^2} \alpha} \sum \theta_j .
\]

This gives the semi-classical approximation of our original formula.

The exact quantum formula can be considered as a result of Sklyanin’s quantisation in separated variables. The function \( Q(\theta) \) satisfy the requirements formulated above. The following formula fixes \( \rho \) in the asymptotics:

\[
\rho = \frac{4\nu}{\sqrt{\pi (\nu - 1)}} \Gamma \left( 1 - \frac{1}{2\nu} \right) \Gamma \left( \frac{1}{2} + \frac{1}{2\nu} \right) \cdot \frac{(\mu \Gamma(\nu))^{\frac{1}{\nu}}}{\sqrt{\nu - 1}} R.
\]

The formula for \( A^{cl} \) is obtained by the steepest descend method.
5. Exact computation in quantum case

We had the equation

\[ Q(\theta + \frac{\pi i}{2})Q(\theta - \frac{\pi i}{2}) - Q(\theta + \frac{\pi i(\nu-2)}{2\nu})Q(\theta - \frac{\pi i(\nu-2)}{2\nu}) = 1, \]

which can be thought about as a discrete Liouville equation. It is convenient to introduce the function \( \epsilon(\theta) \) via

\[ e^{-\epsilon(\theta)} = Q(\theta + \frac{\pi i(\nu-2)}{2\nu})Q(\theta - \frac{\pi i(\nu-2)}{2\nu}). \]

The fundamental role for the computation of the one-point functions is played by the function \( \omega(\theta, \theta') \) which is a "Green function" for the linearised equation with a twist \( \alpha \).
Consider the linear operator:

\[
(D_{\alpha} f) (\theta) = (1 + e^{\epsilon(\theta)}) \left( f(\theta + \frac{\pi i}{2}) + f(\theta - \frac{\pi i}{2}) \right) \\
- e^{\pi i \alpha} f(\theta + \frac{\pi i(\nu - 2)}{2\nu}) - e^{-\pi i \alpha} f(\theta - \frac{\pi i(\nu - 2)}{2\nu}).
\]

The function \(\omega(\theta, \theta')\) satisfies the equations:

\[
D_{\alpha} \omega = \omega D_{-\alpha} = f,
\]

where \(f\) is a "\(\delta\)-function": \(f(\theta - \theta') = \frac{1}{2\pi \cosh(\theta - \theta')}\).

This function allows the asymptotical expansion \(\theta \to \epsilon \infty, \theta' \to \epsilon' \infty\):

\[
\omega(\theta, \theta') \sim \sum_{j,k=1}^{\infty} e^{-\epsilon(2j-1)\theta - \epsilon'(2k-1)\theta'} \omega_{2j-1,2k-1}.
\]
Main conjecture. We claim that

\[
\frac{\langle \Phi_{\alpha-2\nu-1}(0) \rangle}{\langle \Phi_{\alpha}(0) \rangle} = C'(\alpha) \left( 1 + \frac{2 \sin \pi (\alpha + \frac{1}{\nu})}{\pi} \omega_{1,-1} \right),
\]

where

\[
C'(\alpha) = (\mu \Gamma(\nu))^{4x} \frac{\Gamma(-2\nu x)\Gamma(x)\Gamma(1/2 - x)}{\Gamma(2\nu x)\Gamma(-x)\Gamma(x + 1/2)}, \quad x = \frac{\alpha}{2} + \frac{1-\nu}{2\nu}.
\]

Moreover, using \(\omega(\theta, \theta')\) one can write down a formula for the normalised to \(\Phi_{\alpha}\) one-point functions of all the descendants of operators \(\Phi_{\alpha+2m\nu-1}\).
6. Checking classical case

In classical case $\nu = 1$, and $1 + e^{\epsilon(\theta)} = e^{r \cosh \theta}$. Hence

$$
\omega(\theta + \frac{\pi i}{2}, \theta')(1 - e^{-r \cosh \theta - \pi i \alpha}) + \omega(\theta - \frac{\pi i}{2}, \theta')(1 - e^{-r \cosh \theta + \pi i \alpha})
$$

$$
e^{-r \cosh \theta}
= \frac{e^{-r \cosh \theta}}{2\pi \cosh(\theta - \theta')}. 
$$

This can be easily solved. The equivalence to Lukyanov’s formula reduces to the identity

$$
1 - \frac{2 \sin \pi \alpha}{\pi} \omega_{1,-1} = \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \log \left( \frac{1 - e^{-r \cosh \theta + \pi i \alpha}}{1 - e^{-r \cosh \theta - \pi i \alpha}} \right) d\theta \right).
$$

The proof is an exercise on Riemann-Hilbert problem.
7. Comparison with the Liouville three-point function

Conformal limit corresponds to $R \to 0$. In this limit we should be able to make comparison with the Liouville model with the central charge $c = 1 + 6Q^2$, $Q = b + 1/b$.

There is a huge region in the configuration space where only the dynamics of the zero-mode counts. We have the ZZ quantisation condition

$$\frac{2P}{b} \log \left( R^{1+b^2} \mu \Gamma(b^2) \right) = -\frac{\pi}{2} (1 + 2n) + \text{Im} \log \left( \Gamma(1 + 2i P/b) \Gamma(1 + 2i Pb) \right),$$

which defines $P$ as a function of $R$. The relation between shG one-pint functions and Liouville three-point functions reads:

$$\langle \Phi_\alpha(0) \rangle_R \simeq N(P(R)) \cdot C(Q/2 - iP(R), a, Q/2 + iP(R)), \quad R \to 0,$$

(recall $a = Q\alpha/2$).

This can be checked numerically.
8. More mysterious relation to Liouville

Following Al. Zamolodchikov consider the discrete Liouville equation:

$$X(u + 1, v)X(u - 1, v) - X(u, v + 1)X(u, v - 1) = 1.$$  

The continuous case is obtained by $u = m, v = n, x = \Delta m, y = \Delta n,$

$$X(m, n) = \Delta^{-1}e^{-\frac{1}{2}\phi(x,y)}.$$  

Obviously, the equation

$$Q(\theta + \frac{\pi i}{2})Q(\theta - \frac{\pi i}{2}) - Q(\theta + \frac{\pi i(\nu - 2)}{2\nu})Q(\theta - \frac{\pi i(\nu - 2)}{2\nu}) = 1,$$

is obtained by the reduction:

$$X(u, v + 1) = X(u + \frac{\nu - 2}{\nu}, v),$$

rescaling $\theta = \frac{\pi i}{2}u$, and omitting the redundant variable $v$. This relation remains a mystery to which we add one more.
Varying the discrete Liouville equation we obtain the linear operator:

\[(Df)(u, v) = \left(1 + \frac{1}{X(u, v + 1)X(u, v - 1)}\right)(f(u + 1, v) + f(u - 1, v)) - f(u, v + 1) - f(v, v - 1).\]

The operator \(D_\alpha\) which enters the definition of \(\omega(\theta, \theta')\) is obtained by the reduction:

\[f(u, v + 1) = e^{\pi i \alpha} f\left(u + \frac{v - 2}{\nu}, v\right).\]

What is the meaning of all that?
9. Thermodynamic Bethe Ansatz

Using the definition

\[ e^{-\epsilon(\theta)} = Q(\theta + \frac{\pi i (\nu - 2)}{2\nu}) Q(\theta - \frac{\pi i (\nu - 2)}{2\nu}) . \]

rewrite q-Wronskian equation as

\[ Q(\theta + \frac{\pi i}{2}) Q(\theta - \frac{\pi i}{2}) = 1 + e^{-\epsilon(\theta)} . \]

Together with asymptotics and absence of zeros it means

\[ \log Q(\theta) = -\rho \cosh \theta + \int_{-\infty}^{\infty} \frac{1}{2\pi \cosh(\theta - \theta')} \log \left( 1 + e^{-\epsilon(\theta')} \right) d\theta' . \]

This implies the TBA equation

\[ \epsilon(\theta) = 2\pi Rm \cosh \theta - \int_{-\infty}^{\infty} \log \left( 1 + e^{-\epsilon(\theta')} \right) \Phi(\theta - \theta') d\theta' . \]
Kernels:

$$\Phi(\theta, \theta') = \Phi_0(\theta, \theta'),$$

$$\Phi_\alpha(\theta) = \frac{e^{i\pi\alpha}}{2\pi \cosh(\theta + \pi i \frac{\nu - 2}{2\nu})} + \frac{e^{-i\pi\alpha}}{2\pi \cosh(\theta - \pi i \frac{\nu - 2}{2\nu})}.$$ 

Denote by $*$ the convolution with the measure

$$\frac{d\theta}{1 + e^{\epsilon(\theta)}}.$$ 

Then

$$\omega = f \ast f + f \ast R_{\text{dress}, \alpha} \ast f,$$

where the resolvent satisfies the equation

$$R_{\text{dress}, \alpha} - \Phi_\alpha \ast R_{\text{dress}, \alpha} = \Phi_\alpha.$$