On one-point functions for quantum sinh-Gordon model.

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1. Integral representation of the one-point function

My final goal is to compute the following infinite-fold integral:

$$\langle \Phi_\alpha(0) \rangle_R$$

$$= \int \prod_{j=-\infty}^{\infty} d\theta_j \prod_{j=-\infty}^{\infty} Q^2(\theta_j) e^{(\tilde{\nu}+\nu)\alpha\theta_j} \prod_{i<j} \sinh \nu(\theta_i - \theta_j) \sinh \tilde{\nu}(\theta_i - \theta_j) \,,$$

where

$$\frac{1}{\nu} + \frac{1}{\tilde{\nu}} = 1, \quad \nu = 1 + b^2, \;\; \tilde{\nu} = 1 + b^{-2} \,,$$

the function $Q(\theta)$ is defined by three requirements.

- Quantum Wronskian equation

$$Q\big(\theta+\tfrac{\pi i}{2}\big)Q\big(\theta-\tfrac{\pi i}{2}\big)-Q\big(\theta+\tfrac{\pi i(\nu-2)}{2\nu}\big)Q\big(\theta-\tfrac{\pi i(\nu-2)}{2\nu}\big)=1\,.$$

- Asymptotics

$$\log Q(\theta)=-\rho\cosh\theta+O(1),\quad \theta\to\pm\infty\,.$$

- Ground state: no zeros in the strip $|\mathrm{Im}(\theta)|\leq\pi$.

For the final formula to be applicable only the first requirement really count, two others may be eased considerably.

2. Meaning of the integral, semiclassical consideration

Consider the sinh-Gordon model with the action:

$$\mathcal{A}=\int\Big\{\Big[\frac{1}{4\pi}\partial_z\varphi(z,\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z})+\frac{2\boldsymbol{\mu}^2}{\sin\pi b^2}\cosh(b\varphi(z,\bar{z}))\Big\}\frac{idz\wedge d\bar{z}}{2}\,,$$

on the cylinder

$$C=\mathbb{C}/2\pi iR\,\mathbb{Z}\,.$$

Then the main integral gives the SoV representation of the functional integral

$$\langle\Phi_\alpha(0)\rangle_R=\int e^{-\mathcal{A}+a\varphi(0)}\prod_{z,\bar{z}\in C}\mathcal{D}\varphi(z,\bar{z})\,,$$

with the convention

$$a=\frac{1}{2}(b+b^{-1})\alpha\,.$$

For the sake of classical limit $\hbar = b^2 \to 0$ it is convenient to introduce $\phi(z,\bar{z}) = b\varphi(z,\bar{z})$. Then the classical action is

$$\mathcal{A} = \frac{1}{4\pi}\int \left\{\left[\partial_z\phi(z,\bar{z})\partial_{\bar{z}}\phi(z,\bar{z}) + 2m^2\cosh(\phi(z,\bar{z}))\right\}\frac{idz\wedge d\bar{z}}{2}\,,$$

where $m^2 = \boldsymbol{\mu}^2\frac{16\pi b^2}{\sin\pi b^2}$ is semi-classically finite.
The main contribution to the functional integral is given by regularised action evaluated on the classical solution

$$\partial_z\partial_{\bar{z}}\phi^{\mathrm{cl}}(z,\bar{z}) = \frac{m}{2}\sinh\phi^{\mathrm{cl}}(z,\bar{z})\,,$$

rapidly decreasing at infinity, and possessing the singularity at 0:

$$\phi^{\mathrm{cl}}(z,\bar{z}) \simeq 2\alpha\log|z|\,,\quad z\to 0\,.$$

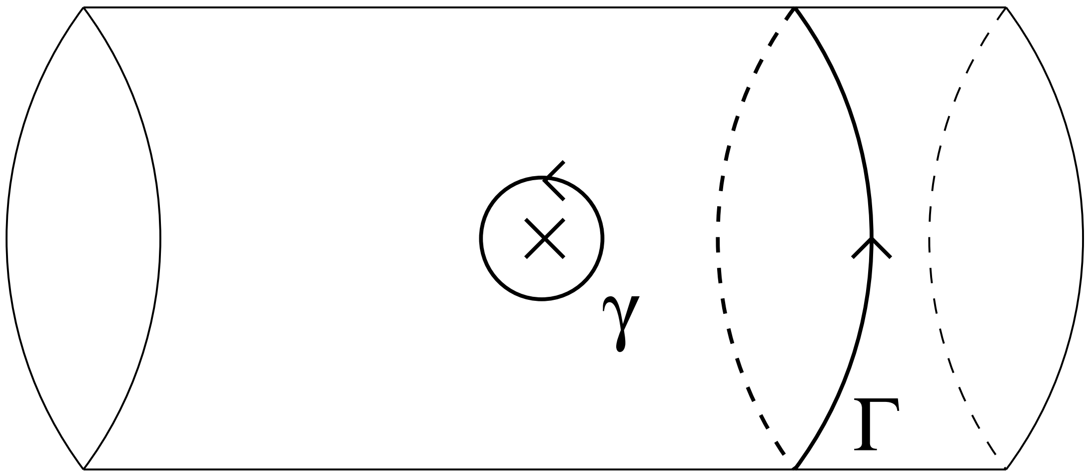
This can be computed (Lukyanov):

$$\mathcal{A}^{\mathrm{cl}} = \frac{\alpha^2}{2}\log\left(\frac{m}{4}\right) + \int_0^{\infty}\frac{dt}{t}\left(\frac{\sinh^2(\alpha t)}{t\sinh(2t)} - \frac{\alpha^2}{2}e^{-2t}\right)$$
$$- \int\limits_0^{\alpha} d\alpha \int\limits_{-\infty}^{\infty} \frac{d\theta}{2\pi i}\log\left(\frac{1-e^{-r\cosh\theta-\pi i\alpha}}{1-e^{-r\cosh\theta+\pi i\alpha}}\right) ,$$

where $r = 2\pi m R$. With this we have:

$$\langle\Phi_\alpha(0)\rangle_R \sim e^{-\frac{1}{b^2}\mathcal{A}^{\mathrm{cl}}} .$$

3. Hamiltonian approach



Consider the Hamiltonian picture with time going along the cylinder, and space along Γ (Matsubara). Then

$$\langle \Phi_\alpha(0) \rangle_R = \langle \Psi | \Phi_\alpha(0) | \Psi \rangle \,,$$

where Ψ is the ground state corresponding to the maximal eigenvalue of the Matsubara transfer-matrix.

4. Classics

Sinh-Gordon equation is equivalent to the zero-curvature condition for the connection:

$$L = \partial_z + \frac{1}{4}\partial_z\phi\sigma^3 - \lambda\frac{m}{2}\left(e^{\phi}\sigma^+ + e^{-\phi}\sigma^-\right) ,$$
$$\overline{L} = \partial_{\bar z} - \frac{1}{4}\partial_{\bar z}\phi\sigma^3 - \frac{1}{\lambda}\frac{m}{2}\left(e^{\phi}\sigma^- + e^{-\phi}\sigma^+\right) .$$

We construct the monodromy matrix:

$$M(\lambda, z) = P\exp\int\limits_{z}^{z+2\pi iR}(Ldz + \overline{L}d\bar z) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} .$$

Its trace $T(\lambda) = \mathrm{Tr}M(\lambda, z)$, does not depend on z, this is the generating function of integrals of motion. The matrix elements of $M(\lambda, z)$ are single-valued functions on the hyper-elliptic curve of infinite genus:

$$T(\lambda) = \mu + \frac{1}{\mu} .$$

Separated variables. Zeros of the function $B(\lambda)$ are real. We order them and denote by λ_j, $-\infty < j < \infty$. These zeros depend on y. They either oscillate inside the zones $|T(\lambda)| > 2$ or stay at double points $|T(\lambda)| = 2$. The variables $\log\lambda_j$ and $\log\mu_j$ are canonical:

$$pdq = \sum_{j=-\infty}^{\infty} \log\mu_j d\log\lambda_j \, .$$

Consider zeros of $T(\lambda)$ (τ_j) as one half of coordinates on the phase space. Then we have for Liouville measure

$$(dp\wedge dq)^{\wedge\frac{\infty}{2}} = \prod_{j=-\infty}^{\infty} \frac{1}{\mu_j - \mu_j^{-1}} \prod_{i<j} (\lambda_i^2 - \lambda_j^2) \bigwedge_{j=-\infty}^{\infty} d\log\lambda_j \bigwedge_{j=-\infty}^{\infty} d\tau_j \, .$$

It is convenient to switch to the variables $\theta_j = \log\lambda_j$.

Now everything is prepared for writing the semi-classical expression for the matrix elements of an operator $\mathcal{O}(\{\theta_j\})$. There is a cohomological argument for considering only this kind of operators.

We have

$$\langle\Psi_1|\mathcal{O}|\Psi_2\rangle = \int \prod_{j=-\infty}^{\infty} d\theta_j \prod_{j=-\infty}^{\infty} Q_1(\theta_j)Q_2(\theta_j)e^{\frac{1}{b^2}2j\theta_j} \prod_{i<j} \sinh(\theta_i-\theta_j)\,,$$

where

$$Q(\theta_j) \simeq \frac{1}{(\mu_j-\mu_j^{-1})^{\frac{1}{2}}} \exp\left\{\frac{1}{ib^2}\int^{\lambda_j} \log\mu \, d\log\lambda\right\}.$$

We consider the simplest case when both $\Psi_{1,2}$ are the ground state. For this case

$$\mu = e^{\frac{1}{2}mR(\lambda-\lambda^{-1})}\,,$$

which implies

$$Q(\theta) \simeq \frac{1}{(\sinh(mR\sinh\theta))^{\frac{1}{2}}} e^{-\frac{1}{b^2}mR\cosh\theta}\,.$$

For the operator $\mathcal{O}$ we take

$$\Phi_\alpha(0) = e^{\frac{1}{b^2}\alpha\sum\theta_j}\,.$$

This gives the semi-classical approximation of our original formula.

The exact quantum formula can be considered as a result of Sklyanin's quantisation in separated variables. The function $Q(\theta)$ satisfy the requirements formulated above. The following formula fixes ρ in the asymptotics:

$$\rho = \frac{4\nu}{\sqrt{\pi(\nu-1)}}\Gamma\left(1-\tfrac{1}{2\nu}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2\nu}\right)\cdot\frac{(\boldsymbol{\mu}\Gamma(\nu))^{\frac{1}{\nu}}}{\sqrt{\nu-1}}R\,.$$

The formula for $\mathcal{A}^{\mathrm{cl}}$ is obtained by the steepest descend method.

5. Exact computation in quantum case

We had the equation

$$Q\big(\theta+\tfrac{\pi i}{2}\big)Q\big(\theta-\tfrac{\pi i}{2}\big)-Q\big(\theta+\tfrac{\pi i(\nu-2)}{2\nu}\big)Q\big(\theta-\tfrac{\pi i(\nu-2)}{2\nu}\big)=1\,,$$

which can be thought about as a discrete Liouville equation.
It is convenient to introduce the function $\epsilon(\theta)$ via

$$e^{-\epsilon(\theta)}=Q\big(\theta+\tfrac{\pi i(\nu-2)}{2\nu}\big)Q\big(\theta-\tfrac{\pi i(\nu-2)}{2\nu}\big)\,.$$

The fundamental role for the computation of the one-point functions is played by the function $\omega(\theta,\theta')$ which is a "Green function" for the linearised equation with a twist α.

Consider the linear operator:

$$\left(\mathcal{D}_\alpha f\right)(\theta) = \left(1 + e^{\epsilon(\theta)}\right)\left(f(\theta + \tfrac{\pi i}{2}) + f(\theta - \tfrac{\pi i}{2})\right)$$
$$- e^{\pi i \alpha} f(\theta + \tfrac{\pi i(\nu-2)}{2\nu}) - e^{-\pi i \alpha} f(\theta - \tfrac{\pi i(\nu-2)}{2\nu})\,.$$

The function $\omega(\theta, \theta')$ satisfies the equations:

$$\mathcal{D}_\alpha \omega = \omega \mathcal{D}_{-\alpha} = f\,,$$

where f is a "δ-function": $f(\theta - \theta') = \frac{1}{2\pi \cosh(\theta - \theta')}$.
This function allows the asymptotical expansion $\theta \to \epsilon\infty$, $\theta' \to \epsilon'\infty$:

$$\omega(\theta, \theta') \simeq \sum_{j,k=1}^{\infty} e^{-\epsilon(2j-1)\theta - \epsilon'(2k-1)\theta'} \omega_{2j-1,2k-1}\,.$$

Main conjecture. We claim that

$$\frac{\langle \Phi_{\alpha-2\frac{\nu-1}{\nu}}(0)\rangle}{\langle \Phi_\alpha(0)\rangle} = C(\alpha)\Big(1+\frac{2\sin\pi(\alpha+\frac{1}{\nu})}{\pi}\omega_{1,-1}\Big)\,,$$

where

$$C(\alpha) = (\boldsymbol{\mu}\Gamma(\nu))^{4x}\ \frac{\Gamma(-2\nu x)\Gamma(x)\Gamma(1/2-x)}{\Gamma(2\nu x)\Gamma(-x)\Gamma(x+1/2)}\,,\quad x=\frac{\alpha}{2}+\frac{1-\nu}{2\nu}\,.$$

Moreover, using $\omega(\theta,\theta')$ one can write down a formula for the normalised to Φ_α one-point functions of all the descendants of operators $\Phi_{\alpha+2m\frac{\nu-1}{\nu}}$.

6. Checking classical case

In classical case $\nu = 1$, and $1 + e^{\epsilon(\theta)} = e^{r\cosh\theta}$. Hence

$$\omega(\theta + \tfrac{\pi i}{2}, \theta')\bigl(1 - e^{-r\cosh\theta - \pi i\alpha}\bigr) + \omega(\theta - \tfrac{\pi i}{2}, \theta')\bigl(1 - e^{-r\cosh\theta + \pi i\alpha}\bigr)$$
$$= \frac{e^{-r\cosh\theta}}{2\pi\cosh(\theta - \theta')} .$$

This can be easily solved. The equivalence to Lukyanov's formula reduces to the identity

$$1 - \frac{2\sin\pi\alpha}{\pi}\omega_{1,-1} = \exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty} \log\left(\frac{1 - e^{-r\cosh\theta + \pi i\alpha}}{1 - e^{-r\cosh\theta - \pi i\alpha}}\right) d\theta\right) .$$

The proof is an exercise on Riemann-Hilbert problem.

7. Comparison with the Liouville three-point function

Conformal limit corresponds to $R \to 0$. In this limit we should be able to make comparison with the Liouville model with the central charge $c = 1 + 6Q^2$, $Q = b + 1/b$.
There is a huge region in the configuration space where only the dynamics of the zero-mode counts. We have the ZZ quantisation condition

$$\frac{2P}{b} \log\left(R^{1+b^2} \boldsymbol{\mu} \Gamma(b^2)\right) = -\frac{\pi}{2}(1 + 2n) + \operatorname{Im} \log\left(\Gamma(1 + 2iP/b)\Gamma(1 + 2iPb)\right) \ ,$$

which defines P as a function of R. The relation between shG one-pint functions and Liouville three-point functions reads:

$$\langle \Phi_\alpha(0) \rangle_R \simeq \mathcal{N}(P(R)) \cdot C(Q/2 - iP(R),\ a,\ Q/2 + iP(R)), \quad R \to 0\,,$$

(recall $a = Q\alpha/2$).

This can be checked numerically.

8. More mysterious relation to Liouville

Following Al. Zamolodchikov consider the discrete Liouville equation:

$$X(u+1,v)X(u-1,v)-X(u,v+1)X(u,v-1)=1\,.$$

The continuous case is obtained by $u=m$, $v=n$, $x=\Delta m$, $y=\Delta n$,

$$X(m,n)=\Delta^{-1}e^{-\frac{1}{2}\phi(x,y)}\,.$$

Obviously, the equation

$$Q\big(\theta+\tfrac{\pi i}{2}\big)Q\big(\theta-\tfrac{\pi i}{2}\big)-Q\big(\theta+\tfrac{\pi i(\nu-2)}{2\nu}\big)Q\big(\theta-\tfrac{\pi i(\nu-2)}{2\nu}\big)=1\,,$$

is obtained by the reduction:

$$X(u,v+1)=X(u+\tfrac{\nu-2}{\nu},v)\,,$$

rescaling $\theta=\frac{\pi i}{2}u$, and omitting the redundant variable v. This relation remains a mystery to which we add one more.

Varying the discrete Liouville equation we obtain the linear opetrator:

$$(\mathcal{D}f)(u,v) = \Big(1 + \frac{1}{X(u,v+1)X(u,v-1)}\Big)(f(u+1,v) + f(u-1,v)) \\ - f(u,v+1) - f(v,v-1)\,.$$

The operator $\mathcal{D}_\alpha$ which enters the definition of $\omega(\theta,\theta')$ is obtained by the reduction:

$$f(u,v+1) = e^{\pi i\alpha} f(u + \tfrac{\nu-2}{\nu}, v)\,.$$

What is the meaning of all that?

9. Thermodynamic Bethe Ansatz

Using the definition

$$e^{-\epsilon(\theta)} = Q\big(\theta + \tfrac{\pi i(\nu-2)}{2\nu}\big)Q\big(\theta - \tfrac{\pi i(\nu-2)}{2\nu}\big) \,.$$

rewrite q-Wronskian equation as

$$Q\big(\theta + \tfrac{\pi i}{2}\big)Q\big(\theta - \tfrac{\pi i}{2}\big) = 1 + e^{-\epsilon(\theta)} \,.$$

Together with asymptotics and absence oz zeros it means

$$\log Q(\theta) = -\rho \cosh\theta + \int\limits_{-\infty}^{\infty} \frac{1}{2\pi \cosh(\theta - \theta')} \log\left(1 + e^{-\epsilon(\theta')}\right) d\theta' \,.$$

This implies the TBA equation

$$\epsilon(\theta) = 2\pi R m \cosh\theta - \int_{-\infty}^{\infty} \log\left(1 + e^{-\epsilon(\theta')}\right) \Phi(\theta - \theta') d\theta' \,.$$

Kernels:

$$\Phi(\theta, \theta') = \Phi_0(\theta, \theta') \,,$$

$$\Phi_\alpha(\theta) = \frac{e^{i\pi\alpha}}{2\pi\cosh(\theta + \pi i \frac{\nu-2}{2\nu})} + \frac{e^{-i\pi\alpha}}{2\pi\cosh(\theta - \pi i \frac{\nu-2}{2\nu})} \,.$$

Denote by $*$ the convolution withe measure

$$\frac{d\theta}{1 + e^{\epsilon(\theta)}} \,.$$

Then

$$\omega = f * f + f * R_{\mathrm{dress},\alpha} * f \,,$$

where the resolvent satisfies the equation

$$R_{\mathrm{dress},\alpha} - \Phi_\alpha * R_{\mathrm{dress},\alpha} = \Phi_\alpha \,.$$