Quantum separation of variables as exact approach to the spectrum and dynamics of integrable quantum models

WORKSHOP ON CORRELATION FUNCTIONS OF QUANTUM INTEGRABLE MODELS, Université de Bourgogne, IMB, Dijon, France, 04-06/09/2013

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Subject related to past and current collaborations with J. Teschner, N. Grosjean, J.-M. Maillet, S. Faldella, N. Kitanine, D. Levy-Bencheton, V. Terras.

Main aims:

- To solve exactly lattice integrable quantum models by quantum separation of variables (SOV) characterizing both their spectrum and dynamics (time dependent correlation functions).
- To define a microscopic approach to solve exactly 1+1 dimensional quantum field theories (QFT) by using the SOV solution of their integrable lattice regularizations.

Plan of the seminar:

- Introduction:
  - Recall of classical integrability, inverse scattering method and separation of variables.
  - Introduction to the corresponding quantum analog concepts:
    - Quantum integrable models.
    - Quantum inverse scattering method (QISM)[1].
    - Quantum separation of variables (SOV)[2].
- Quantum separation of variables implementation for a specific integrable quantum models:
  - SOV for the integrable lattice regularization of sine-Gordon model.
  - Completeness of spectrum description by SOV and brief comment on continuum limit.
  - Matrix elements of local operators, fundamental results toward the full model dynamics.
  - Universal characterization of spectrum&dynamics of integrable quantum models by SOV

[1] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.
[2] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.

SOV approach developed for several fundamental integrable quantum models from the spectrum to the dynamics in the following papers:

G.N., J.Teschner, *The Sine-Gordon model revisited I*, **J. Stat. Mech. 1009 : P09014, 2010**

G.N., *Reconstruction of Baxter Q-operator from Sklyanin SOV for cyclic representations of integrable quantum models*, **Nucl. Phys. B835: 263, 2010**

G.N., *Completeness of Bethe Ansatz by Sklyanin SOV for Cyclic Representations of Integrable Quantum Models*, **JHEP 1103:123,2011**

N.Grosjean, J.-M.Maillet, G.N., *On the form factors of local operators in the lattice sine-Gordon model*, **J.Stat.Mech. (2012) P10006**

N.Grosjean, G.N., *The τ_2-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin's SOV method*, **J. Stat. Mech. (2012) P11005**

G.N., *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*, **Nucl.Phys.B 870: 397 (2013);** *Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables*, **J.Math.Phys. 54, 053516 (2013);** *Antiperiodic dynamical 6-vertex model I: Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic 8-vertex model*, **J. Phys. A: Math. Theor. 46 075003, 2013;** *On the developments of Sklyanin's quantum separation of variables for integrable quantum field theories*, **invited contribution to the Proceedings of the XVIIth INTERNATIONAL CONGRESS ON MATHEMATICAL PHYSICS, August 2012, Aalborg, Danemark**

G.N. *Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and matrix elements of some quasi-local operators*, **J. Stat. Mech. (2012) P10025**

S.Faldella, N.Kitanine, G.N., *Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms*, **arXiv:1307.3960**

S.Faldella, G.N., *SOV approach for integrable quantum models associated to general representations on spin-1/2 chains of the 8-vertex reflection algebra*, **arXiv:1307.5531**

N.Grosjean, J.-M.Maillet, G.N., *On form factors of local operators in the Bazhanov-Stroganov model and the chiral Potts model*, **soon on the archive**

D. Levy-Bencheton, G.N., V. Terras., *Antiperiodic dynamical 6-vertex II: Form factors by separation of variables*, **soon on the archive**

D. Levy-Bencheton, G.N., V. Terras., *8-vertex model: complete spectrum and matrix elements by SOV*, **soon on the archive**

• The statement of universality has been verified by implementing this SOV method for several other fundamental models, like the closed XXZ quantum spin chains with the antiperiodic boundary conditions and the open XXZ and XYZ ones with the most general integrable boundary conditions (describing also systems out of equilibrium like PASEP), the dynamical SOS models with antiperiodic boundaries and the 8-vertex models as well as the 6-vertex models in the most general cyclic representations (Bazhanov-Stroganov-model) and the chiral Potts model (central model in statistical mechanics).
The following important universal features appear:

- the eigenvalue and eigenstates of the Hamiltonian of the model are completely characterized by classifying all the solutions (in a model dependent class of functions) to a given set of differences equations (Baxter's second order difference equations for models associated to $U_q(\hat{sl}(2))$ quantum groups),
- the scalar products for eigenstates have the simple form of determinants and the form factors of local operators (which are the basic building blocks to write any correlation function) admit determinant representation given by simple modifications of the scalar product formula.

It is the relative mathematical simplicity with which one can compute exactly the spectrum and dynamics of quantum many-body systems which cannot be solved by other quantum integrable methods and the universality in the representations of the results which make me hope that the SOV method that I am contributing to develop will allow the solutions of advanced models and will become more and more central in the community of quantum integrability.

Recall of classical integrability and separation of variables

Definition Let us consider a 2N-dimensional symplectic manifold and N independent functions H_j in involution on it w.r.t. the Poisson bracket: $\{H_n, H_m\} = 0 \ \ \forall (n, m) \in \{1, .., \mathsf{N}\}^2$, then each H_j defines the Hamiltonian of a Liouville's integrable system.

Definition Let y_n and p_n be N couples of canonical conjugate variables of the integrable system $\{y_n, y_m\} = \{p_n, p_m\} = 0, \ \{y_n, p_m\} = \delta_{n,m} \ \forall (n, m) \in \{1, .., \mathsf{N}\}^2$, then y_n are separate variables for the integrable system if and only if $\exists$ N separate relations of the form:

$$F_n(y_n, p_n, H_1, ..., H_\mathsf{N}) = 0 \quad \forall n \in \{1, .., \mathsf{N}\}.$$

The principal function S, solution of the Hamilton-Jacobi equation, is then separate in the y_n.

Classical integrability by inverse scattering method If there exist a pair of Lax matrices:

$$\partial M(\lambda)/\partial t = [N(\lambda), M(\lambda)] \leftrightarrow \text{ equation of motion},$$

then the integrals of the motion are defined by the spectral invariants[3] of the matrix $M(\lambda)$, e.g. $\mathsf{T}(\lambda) = \mathrm{Tr}\, M(\lambda)$ are integrals of the motion. Moreover, if $M(\lambda)$ is a 2×2 Lax matrix, then[3] the separate relations are given by the spectral curves:

$$\boxed{\det(M(y_n) - p_n I) = 0, \ \ \text{with } y_n \text{ zeros of the classical } B(\lambda) \equiv (M(\lambda))_{1,2}\,.}$$

[3]E. K. Sklyanin Com. Math. Phys. 150 (1992) 181.

Definition of quantum integrability

- **A quantum model can be defined by:**

  I) Quantum space $\mathcal{H} \longleftrightarrow \mathcal{H}$ is an Hilbert space,

  II) Observables $\mathcal{O} \longleftrightarrow \mathcal{O} \in$ End$(\mathcal{H})$,

  III) Hamiltonian H of the quantum model $\longleftrightarrow$ time evolution operator $e^{-i\,H\theta/\hbar} \in$ End$(\mathcal{H})$.

- **Problems to solve for quantum models:**

  ◦ To compute eigenvalues and eigenvectors of $H \in$ End$(\mathcal{H})$.

  ◦ To compute matrix elements of observables $\mathcal{O} \in$ End$(\mathcal{H})$ on Hamiltonian eigenvectors: $\langle t'|\mathcal{O}|t\rangle$, where $\langle t'| \in \mathcal{H}^*$ is a co-eigenvector and $|t\rangle \in \mathcal{H}$ is an eigenvector of H.

- **Quantum integrability:**

  $\exists\ \mathsf{T}(\lambda) \in \mathrm{End}(\mathcal{H})$: i) $[\mathsf{T}(\lambda),\mathsf{T}(\lambda')]=0 \quad \forall\lambda,\lambda'$, ii) $[\mathsf{T}(\lambda), H]=0 \quad \forall\lambda \in \mathbb{C}$,

  iii) Complete quantum integrability: simplicity (non-degeneracy) of $\mathsf{T}(\lambda)$ spectrum.

  Note that $\mathsf{T}(\lambda) \in$ End$(\mathcal{H})$ defines the one-parameter family of commuting conserved charges.

- **Characterization of integrability by quantum inverse scattering method (QISM)**

Integrable quantum model with Hamiltonian $H \in \mathrm{End}(\mathcal{H})$ characterized by the Lax operator values matrix $\mathsf{M}_a(\lambda) \in \mathrm{End}(\mathbb{C}^M \otimes \mathcal{H})$:

$$\boxed{\begin{array}{c} \underbrace{R_{ab}(\lambda-\mu)\,\mathsf{M}_a(\lambda)\mathsf{M}_b(\mu) = \mathsf{M}_b(\mu)\mathsf{M}_a(\lambda)\,R_{ab}(\lambda-\mu)}_{\text{Yang-Baxter equation}} \\ \\ \underbrace{\mathsf{T}(\lambda) = \mathrm{Tr}\,\mathsf{M}(\lambda)}_{\text{Transfer matrix}},\ [H,\mathsf{T}(\lambda)] = [\mathsf{T}(\mu),\mathsf{T}(\lambda)] = 0. \end{array}}$$

The elements $(\mathsf{M}_a(\lambda))_{i,j} \in \mathrm{End}(\mathcal{H})$ are generators of the Yang-Baxter algebra and Yang-Baxter algebra representation theory is associated to that of quantum (super)groups $U_q(sl(n,m))$, with $n+m=M$.

In the case $M=2$, $\mathsf{M}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$ with $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda) \in \mathrm{End}(\mathcal{H})$, a part the trace the determinant is the other spectral invariant and its quantum analog:

$$q\text{-det}\,\mathsf{M}(\lambda) = A(\lambda)\,D(\lambda/q) - B(\lambda)\,C(\lambda/q)$$

is a central element of the Yang-Baxter algebra associated to $U_q(sl(2))$.

- **Quantum models on one-dimensional lattices with N quantum sites:**

I) The quantum space $\mathcal{H} \equiv \otimes_{n=1}^{\mathsf{N}} \mathcal{H}_{\mathsf{n}}$ of the model is the tensor product of the local quantum spaces $\mathcal{H}_{\mathsf{n}}$ (Hilbert spaces) associated to each site of the lattice.

II) A local operator $\mathcal{O}_n$ is an element of End$(\mathcal{H})$ which acts as the identity on all the local quantum spaces $\mathcal{H}_{\mathsf{m}\neq\mathsf{n}}$ with the only exception of $\mathcal{H}_{\mathsf{n}}$.

- **Integrable quantum models on the lattice:**

The matrix $\mathsf{M}_a(\lambda)$ $\in$End$(\mathbb{C}^M \otimes \mathcal{H})$ is called monodromy matrix and it is defined by the matrix product of local Lax matrices themselves solutions of the Yang-Baxter equations:

$$\mathsf{M}_0(\lambda) = \mathsf{L}_{0,\mathsf{N}}(\lambda) \cdots \mathsf{L}_{0,1}(\lambda) \quad \text{with } \; \mathsf{L}_{0,n}(\lambda) \in \text{End}(\mathbb{C}^M \times \mathcal{H}_n).$$

- **Motivations to use integrable lattice regularizations for QFT:**

a) To simplify the structure of the quantum space $\mathcal{H}$ (e.g. the quantum space is of finite dimension in the model considered in this seminar) which allows to simplify the solution of the spectral problem.

b) To solve the problem of the identification of the local fields writing them in terms of the generators of the Yang-Baxter algebra thanks to the solution of the quantum inverse problem.

c) To introduce the quantum field theory by some well defined limits starting from the exact and complete characterization of the spectrum and the dynamics of the lattice model.

Quantum separation of variables (SOV): introduction

- Let $\mathsf{Y}_n \in \mathsf{End}(\mathcal{H})$ and $P_n \in \mathsf{End}(\mathcal{H})$ be N couples of canonical conjugate operators:

$$[\mathsf{Y}_n, \mathsf{Y}_m] = [P_n, P_m] = 0, \qquad [\mathsf{Y}_n, P_m] = \delta_{n,m}/2\pi i \;\; \forall (n,m) \in \{1,..,\mathsf{N}\}^2,$$

where $\{\mathsf{Y}_1, ..., \mathsf{Y}_\mathsf{N}\}$ are simultaneous diagonalizable operators with simple spectrum.

Definition: The Y_n define the quantum separate variables for the spectral problem of $\mathsf{T}(\lambda)$ if and only if $\exists$N separate relations of the form:

$$F_n(\mathsf{Y}_n, P_n, \mathsf{T}(\mathsf{Y}_n)) = 0 \quad \forall n \in \{1,..,\mathsf{N}\};$$

quantum analogue of the classical ones in the Hamilton-Jacobi's approach.

- The separate relations are used to solve the spectral problem of $\mathsf{T}(\lambda)$:

$$F_n(y_n, \frac{i}{2\pi}\frac{\partial}{\partial y_n}, t(y_n))\Psi_t(y_1, ..., y_\mathsf{N}) = 0, \quad \text{with } \Psi_t(y_1, ..., y_\mathsf{N}) = \langle y_1, ..., y_\mathsf{N}|t\rangle,$$

where y_n, $t(\lambda)$ and $\langle y_1, ..., y_\mathsf{N}|$, $|t\rangle$ are eigenvalues and eigenstates of Y_n and $\mathsf{T}(\lambda)$, respectively, then:

$$|t\rangle = \sum\nolimits_{\text{over spectrum of } \{\mathsf{Y}_n\}} \Psi_t(y_1, ..., y_\mathsf{N})|y_1, ..., y_\mathsf{N}\rangle$$

$$\mathsf{SOV} \Longrightarrow \Psi_t(y_1, ..., y_\mathsf{N}) = \prod\nolimits_{n=1}^{\mathsf{N}} \mathsf{Q}_t^{(n)}(y_n).$$

where $Q_t^{(n)}(\lambda)$ is a solution of the separate equation in y_n.

Quantum separation of variables (SOV): spectrum characterization

- **Sklyanin's SOV method:** If the monodromy matrix $\mathsf{M}_a(\lambda)$ belongs to End$(\mathbb{C}^{M=2}\otimes\mathcal{H})$ and $B(\lambda)=(M_a(\lambda))_{1,2}$ is a one parameter family of simulteneously diagonalizable operators with simple spectrum, then the Y_n, operators zeros of $B(\lambda)$:
  $\mathsf{Y}_n|y_1,\ldots,y_N\rangle = y_n|y_1,\ldots,y_N\rangle \rightarrow B(y_n)|y_1,\ldots,y_N\rangle = 0,\ \forall n\in\{1,\ldots,N\},$
  are quantum separate variables for the spectral problem of $\mathsf{T}(\lambda)=A(\lambda)+D(\lambda)$:

  $$\mathsf{T}(\lambda)\,|t\rangle = t(\lambda)|t\rangle,\quad |t\rangle \text{ eigenvector of } \mathsf{T}(\lambda),\ t(\lambda) \text{ eigenvalue of } \mathsf{T}(\lambda),$$
  $$\Updownarrow$$
  SOV representation: $|t\rangle=\sum_{\{y\}}\prod_{j=1}^{N}Q_t(y_j)|y_1,\ldots,y_N\rangle,\qquad Q_t(y_j)\in\mathbb{C},$

  Baxter's equation: $t(y_j)Q_t(y_j)=a(y_j)Q_t(y_j/q)+d(y_j)Q_t(y_jq),\quad q\in\mathbb{C}.$

  The Baxter's equation is a direct consequence of the Yang-Baxter equation and it is the quantum analogue of the spectral curve computed in the zeros of the classical $B(\lambda)$.

- **Motivations to use quantum separation of variables (SOV):**

  The SOV method allows to solve the problems which appear in other more traditional methods, like Bethe ansatz and Baxter's Q-operator, giving:
  a) the proof of completeness of the spectrum description,
  b) the analysis of a larger class of integrable quantum models,
  c) more symmetrical approach to classical and quantum integrability.

Integrable lattice regularization of the sine-Gordon model: definitions

- **The classical sine-Gordon (sG) model**: $H = \int_0^R \frac{dx}{4\pi}\left(\Pi^2 + (\partial_x\phi)^2 + 8\pi\mu\cos(2\beta\phi)\right)$, where $\phi(x,t)$ is the sine-Gordon field and $\Pi(x,t)$ is the conjugate momentum:

$$\boxed{\phi : (x,t) \in [0,R] \times \mathbb{R} \to \phi(x,t) \in \mathbb{R},\ \phi(x,t) = \phi(x+R,t),\ \{\Pi(x,t), \phi(y,t)\} = 2\pi\delta(x-y).}$$

- **QISM characterization of lattice sG-model by Lax matrices** (Faddeev, Sklyanin, Takhtajan 1980):

$$\mathsf{M}_0(\lambda) = \mathsf{L}_{0,\mathsf{N}}(\lambda)\cdots\mathsf{L}_{0,1}(\lambda), \quad \mathsf{L}_{0,n}(\lambda) = \begin{pmatrix} \mathsf{U}_n(q^{-1/2}\mathsf{V}_n\kappa_n^2 + q^{1/2}\mathsf{V}_n^{-1}) & (\lambda\mathsf{V}_n/\xi_n - (\lambda\mathsf{V}_n/\xi_n)^{-1})\kappa_n/i \\ (\lambda/\xi_n\mathsf{V}_n - \mathsf{V}_n\xi_n/\lambda)\kappa_n/i & \mathsf{U}_n^{-1}(q^{1/2}\mathsf{V}_n + q^{-1/2}\mathsf{V}_n^{-1}\kappa_n^2) \end{pmatrix}_0.$$

Local fields: $\mathsf{U}_n \equiv e^{i\beta\Pi_n/2}$, $\mathsf{V}_n \equiv e^{-i\beta\phi_n}$, $\mathsf{U}_n\mathsf{V}_m = q^{\delta_{n,m}}\mathsf{V}_m\mathsf{U}_n$, with $q = e^{-i\pi\beta^2}$.
(Weyl algebras)

- **Cyclic representations of local Weyl algebras**: (U_n^p, V_n^p centrals if $\beta^2 = 2p'/p$, p odd)

U_n and V_n are unitary operators for which we can fix $\mathsf{U}_n^p = \mathsf{V}_n^p = 1$ and then we can introduced the following p-dimensional representations for each local Weyl algebra:

$$\mathbf{V}_n|k,n\rangle = q^k|k,n\rangle \quad \mathbf{U}_n|k,n\rangle = |k+1,n\rangle \quad \forall k \in \{1,\ldots,p\}$$

by the action on an eigenbasis of V_n with the cyclicity conditions: $|k+p,n\rangle = |k,n\rangle$.

Note that each local quantum space $\mathcal{H}_\mathsf{n}$ is then p-dimensional and the entire quantum space $\mathcal{H}$ of the representation is a p^N-dimensional Hilbert space.

Integrable lattice regularization of the sine-Gordon model: SOV-representations

- SOV-representation is defined in the basis formed by the B-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_{\mathsf{N}}|\}$ parametrized by the zeros of $B(\lambda)$:

$$\langle \mathbf{y}|\, B(\lambda) = \, b_{\mathbf{y}}(\lambda)\, \langle \mathbf{y}|\,, \qquad b_{\mathbf{y}}(\lambda) \,\equiv\, \prod_{n=1}^{\mathsf{N}} \frac{\kappa_n}{i} \left(\lambda/y_n - y_n/\lambda\right)\,.$$

By the polynomiality w.r.t. λ and the Yang-Baxter commutations relations, it holds:

$$\langle \mathbf{y}|A(\lambda) \;=\; \sum_{n=1}^{\mathsf{N}} \prod_{b\neq n} \tfrac{\lambda/y_b - y_b/\lambda}{y_n/y_b - y_b/y_n}\, a(y_n)\, \langle y_1, \ldots, y_n/q, \ldots, y_{\mathsf{N}}|\,,$$

$$\langle \mathbf{y}|D(\lambda) \;=\; \sum_{n=1}^{\mathsf{N}} \prod_{b\neq n} \tfrac{\lambda/y_b - y_b/\lambda}{y_n/y_b - y_b/y_n}\, d(y_n)\, \langle y_1, \ldots, q\, y_n, \ldots, y_{\mathsf{N}}|\,,$$

$$d(\lambda/q) \equiv q^{\mathsf{N}} a(-\lambda), \quad a(\lambda) \equiv \prod_{r=1}^{\mathsf{N}} \tfrac{\kappa_r \xi_r}{i\lambda}(1 + i q^{-\frac{1}{2}} \tfrac{\lambda \kappa_r}{\xi_r})(1 + i q^{-\frac{1}{2}} \tfrac{\lambda}{\kappa_r \xi_r}).$$

- Eigenvalues $t(\lambda)$ and wave functions $\Psi_t(y_1, \ldots, y_{\mathsf{N}}) \equiv \langle y_1, \ldots, y_{\mathsf{N}}|t\rangle$ are characterized by:

$$a(y_k)\Psi_t(y_1, .., y_k q^{-1}, .., y_{\mathsf{N}}) + d(y_k)\Psi_t(y_1, .., y_k q, .., y_{\mathsf{N}}) = t(y_k)\Psi_t(y_1, .., y_{\mathsf{N}}).$$

These equations follow by computing the matrix elements $\langle y_1, \ldots, y_{\mathsf{N}}|\mathsf{T}(y_k)|t\rangle$ and lead to the factorized ansatz:

$$\Psi_t(y_1, \ldots, y_{\mathsf{N}}) = \prod_{j=1}^{N} Q_t(y_j),$$

where $Q_t(\lambda)$ is a solution of the Baxter equation.

Integrable lattice regularization of the sine-Gordon model: SOV-dates by cyclicity

Definition Average of the family $X(\lambda)$ of commuting operators:

$$\boxed{\mathcal{X}(\Lambda) = \prod_{a=1}^{p} X(\lambda q^{a}), \quad \Lambda = \lambda^{p}.}$$

Proposition
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∘ The averages of Yang-Baxter generators are central and characterized by:

$$\frac{\mathsf{F}(-\Lambda) + \varepsilon \mathsf{F}(\Lambda)}{2} = \begin{cases} \mathcal{B}(\Lambda) \equiv \mathcal{C}(\Lambda) & \text{for } \varepsilon = -1, \\ \mathcal{A}(\Lambda) \equiv \mathcal{D}(\Lambda) & \text{for } \varepsilon = 1, \end{cases}$$

$$\mathsf{F}(\Lambda) \equiv \prod_{a=1}^{p} a(\lambda q^{a}) \equiv \prod_{r=1}^{\mathsf{N}} \frac{\kappa_r^p \xi_r^p}{i^p \Lambda}(1 + \Lambda \frac{(iq^{-\frac{1}{2}}\kappa_r)^p}{\xi_r^p})(1 + \Lambda \frac{(iq^{-\frac{1}{2}})^p}{\kappa_r^p \xi_r^p}).$$

∘ The SOV-dates are fixed in terms of the zeros Z_n of the B-average $\mathcal{B}(\Lambda)$:

$$\boxed{\mathsf{Y}_n^p \equiv Z_n \in \mathbb{C}.}$$

Here we fix the parametrization of the spectrum of the quantum separate variables Y_n by setting:

$$y_n^{(h)} \equiv y_n^{(0)} q^h \in \mathbb{C} \quad \forall\, h \in \{0, \ldots, p-1\}, \forall\, n \in \{1, \ldots, \mathsf{N}\}$$

where $y_n^{(0)}$ is given p-root of Z_n.

Integrable lattice regularization of the sG-model: spectrum and QFT characterization

Let $\beta^2 \equiv p'/p$ be rational, then:

1) **Theorem** **a)** The spectrum of $\mathsf{T}(\lambda)$ is simple for general values of the parameters κ, ξ.

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b) The set Σ_{T} of the eigenvalues of $\mathsf{T}(\lambda)$ coincides with the set of $t(\lambda)$:

$$t(\lambda)Q_t(\lambda) = a(\lambda)Q_t(\lambda q^{-1}) + d(\lambda)Q_t(\lambda q), \quad d(\lambda) \equiv q^{\mathsf{N}}a(-q\lambda)$$

$$\lambda^{(\mathsf{N}-1)}t(\lambda) \in \mathbb{R}[\lambda^2]_{\mathsf{N}-1}\,, Q_t(\lambda) \in \mathbb{R}[\lambda]_{\mathsf{N}(p-1)},$$

$\mathbb{R}[\lambda]_M$ the linear space of *real* polynomials of degree $\leq M$ in λ.

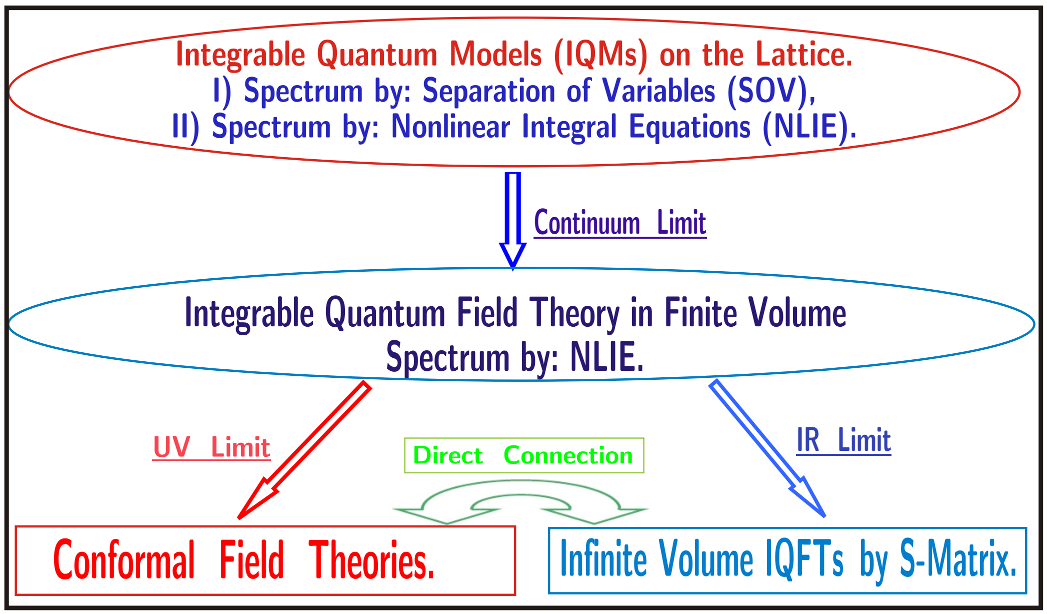
c) Let $t(\lambda) \in \Sigma_{\mathsf{T}}$ then: $\langle y_1^{(h_1)}, \ldots, y_{\mathsf{N}}^{(h_{\mathsf{N}})}|t\rangle \equiv \prod_{k=1}^{N} Q_t(y_k^{(h_k)})$.

- From integrable lattice regularization to QFT:

2) Reformulation of the spectrum in terms of nonlinear integral equations (of thermodynamical Bethe ansatz type) and definition of finite volume quantum field theories by continuum limit.

3) Derivation of Zamolodchikov's S matrix particle description of the sG-QFT spectrum in the infinite volume, (IR) limit.

4) Derivation of the renormalization group fixed point conformal spectrum in the UV limit.



Integrable lattice regularization of the sG-model: dynamics

Definition Form factors $\langle t'|\mathcal{O}_n|t\rangle$ are the matrix elements of a local operator $\mathcal{O}_n$ between the eigencovector $\langle t'|$ and the eigenvector $|t\rangle$ of $\mathsf{T}(\lambda)$.

The form factors are the "elementary objects" w.r.t. any time dependent correlation function can be expanded by using the decomposition of the identity in the transfer matrix eigenbasis:

$$\langle t'|\mathcal{O}_n(\theta)\mathcal{O}_m(\theta)|t''\rangle = \sum_{t\in\Sigma_{\mathsf{T}}} e^{(h_{t'}-h_{t''})\theta}\frac{\langle t'|\mathcal{O}_n|t\rangle\langle t|\mathcal{O}_m|t''\rangle}{\langle t|t\rangle}, \qquad \forall n < m \in \{1, ..., \mathsf{N}\},$$

where $h_{t'}$ and $h_{t''}$ are the Hamiltonian eigenvalues on the eigenstates $|t''\rangle$ and $|t'\rangle$ and by definition of time evolution operator, it holds $\mathcal{O}_n(\theta) \equiv e^{iH\theta}\mathcal{O}_n e^{-iH\theta}$.

Two difficult problems to solve:

i) Reconstruction of local operators $\mathcal{O} \in \mathrm{End}(\mathcal{H})$ in terms of the quantum separate variables.

$\hookrightarrow$ Computation of the action of local operators $\mathcal{O}$ on the eigenvector $|t\rangle$.

ii) Scalar product $\langle t'|t\rangle$ under the form of determinant.

Steps i) and ii) allow us to get in a determinant form the form factors of a basis of local operators.

Method toward dynamics: I) solution of the quantum inverse problem

Proposition The local operator U_1 admits the reconstructions:

T. Oota

$$\mathsf{U}_1 = B^{-1}(\mu_{1,+})A(\mu_{1,+}) = D^{-1}(\mu_{1,+})C(\mu_{1,+}),$$

where $\mu_{1,\pm} \equiv i\kappa_1^{\pm 1} q^{1/2}\xi_1$ are the zeros of the quantum determinant.

∘ Oota's formulae allow to reconstruct all the U_1^k but they do not give a direct reconstruction of the local operators V_1^k.

Proposition In the cyclic representations of the Sine-Gordon model, the local operators V_1^{2k} admit the following reconstruction:

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$$\mathsf{V}_1^{2\mathbf{k}} = \frac{(-\mathbf{1})^{\mathbf{k}}\left(\kappa_1^{2\mathbf{p}}+\mathbf{1}\right)}{\mathbf{p}\kappa_1^{2\mathbf{k}}(\kappa_1^{2}-\kappa_1^{-2})}\sum_{\mathbf{a}=0}^{\mathbf{p}-1}\mathbf{q}^{-\mathbf{k}(2\mathbf{a}-1)}\beta_{\mathbf{a}}, \qquad \text{for } k \in \{1, \dots, p-1\},$$

$$\beta_k = \left(B^{-1}(\mu_{1,+})A(\mu_{1,+})\right)^k A^{-1}(\mu_{1,-})B(\mu_{1,-})\left(B^{-1}(\mu_{1,+})A(\mu_{1,+})\right)^{1-k}.$$

∘ Complete solution of inverse problem for local operators: the above Proposition together with the Oota's formulae allow the reconstruction of all local operators.

Method toward dynamics: II) scalar product formulae

Proposition "Sklyanin's Measure" Right and left B-eigenbasis have the following scalar products:

$$\langle y_1^{(h_1)}, \dots, y_{\mathsf{N}}^{(h_{\mathsf{N}})} | y_1^{(k_1)}, \dots, y_{\mathsf{N}}^{(k_{\mathsf{N}})} \rangle = \prod_{a=1}^{\mathsf{N}} (y_a^{(h_a)})^{\mathsf{N}-1} \delta_{h_a,k_a} \prod_{1\le a<b\le \mathsf{N}} ((y_a^{(h_a)})^2 - (y_b^{(h_b)})^2)^{-1}.$$

Proposition Grosjean, Maillet, Niccoli Let us consider the following two "separated" left and right states:

$$\langle \alpha | = \sum_{h_1,\dots,h_{\mathsf{N}}=1}^{p} \prod_{a=1}^{\mathsf{N}} \alpha_a(y_a^{(h_a)}) \prod_{1\le a<b\le \mathsf{N}} ((y_a^{(h_a)})^2 - (y_b^{(h_b)})^2) \frac{| y_1^{(h_1)}, \dots, y_{\mathsf{N}}^{(h_{\mathsf{N}})} \rangle}{\prod_{b=1}^{\mathsf{N}} (y_a^{(h_a)})^{\mathsf{N}-1}},$$

$$| \beta \rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=1}^{p} \prod_{a=1}^{\mathsf{N}} \beta_a(y_b^{(h_a)}) \prod_{1\le a<b\le \mathsf{N}} ((y_a^{(h_a)})^2 - (y_b^{(h_b)})^2) \frac{\langle y_1^{(h_1)}, \dots, y_{\mathsf{N}}^{(h_{\mathsf{N}})} |}{\prod_{b=1}^{\mathsf{N}} (y_a^{(h_a)})^{\mathsf{N}-1}},$$

then their scalar product has the following determinant form:

$$\langle \alpha | \beta \rangle = \det_{\mathsf{N}} || \mathcal{M}_{a,b}^{(\alpha,\beta)} ||, \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv \sum_{h=1}^{p} \alpha_a(y_a^{(h)}) \beta_a(y_a^{(h)}) (y_a^{(h)})^{2b-(\mathsf{N}+1)}.$$

Moreover, these formulae apply also to the right and left T-eigenstates $\langle t|$ and $|t\rangle$ given for $\alpha_a(y_a^{(h_a)}) = (y_a^{(h_a)})^{\mathsf{N}} Q_t(-y_a^{(h_a)})$ and $\beta_a(y_a^{(h_a)}) = Q_t(y_a^{(h_a)})$, where $Q_t(\lambda)$ is the only solution of the Baxter equation:

$$t(\lambda) Q_t(\lambda) = a(\lambda) Q_t(\lambda q^{-1}) + d(\lambda) Q_t(\lambda q),$$

associated to $t(\lambda) \in \Sigma_{\mathsf{T}}$ as defined in Theorem Niccoli, Teschner 2010.

Integrable lattice regularization of the sG-model: first results on dynamics

Theorem There exists a basis $\mathbb{B}_{\mathcal{H}}$ in End$(\mathcal{H})$ such that for any O $\in \mathbb{B}_{\mathcal{H}}$ the matrix

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elements on the transfer matrix eigenstates read:

$$\langle t'|\mathsf{O}|t\rangle = \det_{\mathsf{N}} ||\Phi_{a,b}^{(\mathsf{O},t',t)}||, \quad \Phi_{a,b}^{(\mathsf{O},t',t)} \equiv \sum_{c=1}^{p} F_{\mathsf{O},b}(y_a^{(c)}) Q_t(y_a^{(c)}) Q_{t'}(-y_a^{(c)})(y_a^{(c)})^{2b-1},$$

where the coefficients $F_{\mathsf{O},b}(y_a^{(c)})$ characterize the operator O.

Let us show two examples:

a) If O is the identity operator, it holds $F_{\mathsf{O},b}(y_a^{(c)}) = 1$ for any $a,\ b \in \{1, \dots, \mathsf{N}\}$.

b) If O $\equiv$ U_1 is the Weyl algebra local generator in site 1, it holds:

$$\begin{aligned} F_{\mathsf{U}_1,b}(y_a^{(c)}) &= y_a^{(c)} \qquad \forall b \in \{1, \dots, \mathsf{N}-1\}, \ \ \forall a \in \{1, \dots, \mathsf{N}\}, \\ F_{\mathsf{U}_1,\mathsf{N}}(y_a^{(c)}) &= \frac{(y_a^{(0)})^{2(\mathsf{N}-1)} q^{1/2} \xi_1 q^{(c+1)(\mathsf{N}-1)} Q_{t'}(-y_a^{(c+1)})}{\prod_{n=2}^{\mathsf{N}} \kappa_n / i(q(\xi_1\kappa_1)^2 + (y_a^{(c+1)})^2) Q_{t'}(-y_a^{(c)})} a(y_a^{(c+1)}). \end{aligned}$$

Main Result: **Universal characterization of spectrum and dynamics by SOV-method.**

Related projects: There are two lines of research that I am developing simultaneously in the use of the SOV method for the exact spectrum and dynamics characterization.

I) To complete the exact characterization of the dynamics for the models already analyzed: Computation of correlation functions.

Research Groups:	Projects:
• J.-M.Maillet et al (ENS, Lyon, France) and B.McCoy (YITP, New-York, USA).	Antiperiodic XXZ spin chains, sine-Gordon model Bazhanov-Stroganov and chiral Potts models.
• V.Terras et al. (ENS, Lyon, France)	The dynamical 6-vertex and 8-vertex models.
• N.Kitanine et al. (IMB, Dijon, France)	Open XXZ and XYZ quantum chains with the most general integrable boundaries.

II) Analysis of spectrum and dynamics by SOV of more advanced integrable quantum models:

• J.-M.Maillet et al (ENS, Lyon, France) J.Teschner (DESY, Hamburg, Germany)	Spin chains associated to higher rank (super)quantum groups spectrum and dynamics, toward solution of nonlinear sigma models by SOV-method.

These last models should lead to the SOV tools for the solution of the Hubbard model.