Form factor approach to the correlation functions of critical models.

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Form factor approach to the asymptotic behavior of correlation functions in critical models, N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. Slavnov and V. Terras, J. Stat. Mech. (2011).

Form factor approach to dynamical correlation functions in critical models, N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. Slavnov and V. Terras, J. Stat. Mech. (2012).

Long-distance asymptotic behavior of multi-point correlation functions in massless quantum integrable models, N. Kitanine, K. K. Kozlowski, J. M. Maillet and V. Terras, to appear, (2013).

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Outline

1 Motivations, results
- Integrable models of interest
- A few predictions

2 Results following from our form factor approach
- The large-distance asymptotics
- The large-distance and long-time asymptotics
- The edge exponents

3 A short sketch of the method
- Around form factor expansion
- Large volume behavior of form factors
- Form factors series and asymptotics

4 Conclusion

Generalities about lattice models

- Linear operator $\mathcal{H}$ on Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_L$.
- Spaces $\mathscr{H}_\ell$ can be finite or infinite dimensional. Often isomophic $\mathscr{H}_\ell \simeq \mathscr{H}_0$.
- Basis of operators $O^{(\alpha)}$ on $\mathscr{H}_0$ $\rightsquigarrow$ local operators $O_\ell^{(\alpha)} = \underbrace{\mathrm{id} \otimes \ldots \mathrm{id}}_{\ell-1 \mathrm{times}} \otimes O^{(\alpha)} \otimes \underbrace{\mathrm{id} \ldots \mathrm{id}}_{N-\ell-1}$.

Often $\mathcal{H}$ has nearest neighbor coupling structure

$$\mathcal{H} = \sum_{j=1}^{L} f(O_j^{(\alpha)}, O_{j+1}^{(\beta)}) + \mathrm{bdry \ \ terms}$$

What one would like to compute?

i) Find the Eigenstates and Eigenvectors of $\mathcal{H} | \Psi_\beta \rangle = E_\beta | \Psi_\beta \rangle$;

ii) Compute in closed form and characterize the correlation functions

$$\langle \Psi_\gamma | O_1^{(\alpha_1)} \ldots O_m^{(\alpha_m)} | \Psi_\beta \rangle \quad ;$$

- Characterize intrinsic & response properties of the system.
- Appear in perturbative expansions: $\mathcal{H} \hookrightarrow \mathcal{H} + \mathcal{H}_{\mathrm{pert}}$.

iii) Characterize the behavior at finite temperature

$$\langle O_m^{(\alpha_m)} O_1^{(\alpha_1)} \rangle_T \equiv \mathrm{tr}\big[\mathrm{e}^{-\frac{\mathcal{H}}{T}} O_m^{(\alpha_m)} O_1^{(\alpha_1)}\big] / \mathrm{tr}\big[\mathrm{e}^{-\frac{\mathcal{H}}{T}}\big]$$

- Program *i)* – *iii)* Get the $L \to +\infty$ limit for critical models and compare with CFT.

Some integrable models

- The XXZ spin-1/2 chain

$$\mathcal{H}_{XXZ} = J \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z + h \sigma_n^z \right\} \quad , \quad \sigma_{n+L} \equiv \sigma_n$$

L: length of circle, Δ anisotropy parameter, $h > 0$ magnetic field.

  - Coordinate Bethe Ansatz for the XXX chain $\Delta = 1$ (**'31** Bethe)

- The non-linear Schrödinger model

$$H = \int_0^L \left\{ \partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y) \Psi^\dagger(y) \Psi(y) \Psi(y) - h \Psi^\dagger(y) \Psi(y) \right\} \mathrm{d}y$$

L: length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

  - Eigenfunctions and spectrum (**'63** Lieb, Liniger).

$$\mathrm{e}^{iL\lambda_j} = \prod_{\substack{a=1 \\ \neq j}}^{N} \frac{\lambda_j - \lambda_a + ic}{\lambda_j - \lambda_a - ic} \qquad \text{so that} \qquad H |\{\lambda_j\}\rangle = \left(\sum_{k=1}^{N} \lambda_k^2 - h \right) |\{\lambda_j\}\rangle$$

Low-lying excitations in 1D quantum Hamiltonians

- 1D *gapless* models at $T = 0K$ are critical
  - **'70** Polyakov Conformal invariance of correlation functions in long-distance regime ;
  - **'84** Cardy Central charge ⇝ finite-size corrections to ground state energy ;

$$E_{G.S.} = L\varepsilon - c\frac{\pi v_F}{6L} + \mathrm{O}\left(\frac{1}{L^2}\right) \quad \text{and} \quad E_{\mathrm{ex}} - E_{G.S.} = \frac{2\pi v_F}{L}\delta$$

  - Bethe Ansatz ⇝ spectrum given by solutions to algebraic equations

$$F(\lambda_j) = \prod_{a=1}^{N} S(\lambda_j, \lambda_k) \quad \text{and} \quad E(\{\lambda_j\}) = \sum_{j=1}^{N} \varepsilon_0(\lambda_j)$$

  - Methods for computing finite-size corrections from Bethe Ansatz
    '87-'95 (Batchelor, Destri, DeVega, Klumper, Pearce, Woynarowich, Zittartz , ...) ;

- Proof of Cardy's predictions for the conformal structure of spectrum:

$$c = 1 \qquad \delta = \left(\frac{N_1}{2\mathcal{Z}}\right)^2 + (\mathcal{Z}N_2)^2 + N_3 \quad \text{and} \quad \text{linear integral equations} \rightsquigarrow v_F\,,\ \mathcal{Z}$$

Asymptotic behavior of correlation functions

- ♦ Critical model $\rightsquigarrow$ algebraic in distance decay of correlators.
  - ★ **'75** Luther, Peschel , **'81** Haldane Luttinger liquid approach to asymptotics ;
  - ★ **'84** Cardy Central charge, scaling dimensions $\rightsquigarrow$ CFT approach to asymptotics;

$\Rightarrow$ Predictions of critical exponents by correspondence with Luttinger liquid/CFT.

- ♦ NLSM ≡ quantum critical model at $T = 0K$ $\rightsquigarrow$ density operator $j(x) = \Psi^\dagger(x)\Psi(x)$

$$\frac{\langle G.S.|j(x)\,j(0)\,|G.S.\rangle}{\langle G.S.\,|\,G.S.\rangle} = \langle j(x)\,j(0)\rangle \simeq \langle j(0)\rangle^2 + \frac{C_1}{x^2} + C_2\frac{\cos(2xp_F)}{x^{2\mathcal{Z}^2}} + \dots$$

and $$\langle \Psi(x)\,\Psi^\dagger(0)\rangle \simeq C_3\, x^{-\frac{1}{2\mathcal{Z}^2}} + \dots$$

No access to non universal constants C_k.

Indirect conjecture for C_k in XXZ at zero magnetic field **'99** Lukyanov , **'03** Lukyanov,Terras .

Turning the time on

- Predictions for the long-distance/long-time behavior at $T = 0K$ restricted to $x \gg v_F t$:

$$\left\langle j(x,t)\, j(0,0) \right\rangle \simeq \langle j(0,0)\rangle^2 + C_1' \frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2} + C_2' \frac{\cos(2xp_F)}{\left(x^2 - v_F^2 x^2\right)^{\mathcal{Z}^2}} + \dots$$

$\Rightarrow$ *Consistency problem* with time-dependent asymptotics

$$\frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2}\left(1 + \mathrm{o}(1)\right) = \frac{1}{x^2}\left(1 + \mathrm{o}(1)\right) \qquad \text{when } \; x \gg v_F t$$

- What happens when x and $v_F t$ are of the same order asymptotically?

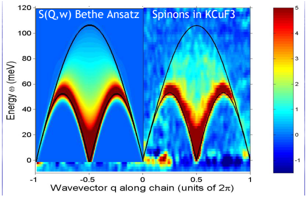
The edge exponents for dynamical structure factors

- Experiments measure dynamical structure factors (Fourier transforms)

$$S(k,\omega) = \int_{\mathbb{R}^2} e^{i(\omega t - kx)} \langle j(x,t)\, j(0,0)\rangle \mathrm{d}x\mathrm{d}t$$

⇝ DSF measured by Fourier sampling of time of flight images or Bragg spectroscopy.

★ **'06** (Caux, Calabrese) Density structure factor in NLSM
★ **'05** (Caux, Hagemans, M.) Density structure factor in XXZ



$S(Q,\omega)$ is the dynamical spin-spin structure factor. The Bethe ansatz curve is computed for a chain of 500 sites and compared to the experimental curve obtained by A. Tennant in Berlin by neutron scattering experiments. Colors indicate the value of the function $S(Q,\omega)$.

Predictions for the behavior near the edges

- ⋆ **'67** (Mahan), **'67** (Nozières, De Dominicis) Arguments for a power-law behavior near edges.
- ⋆ **'08** (Glazman, Imambekov) Non-linear Luttinger liquid $\rightsquigarrow$ predictions for edge exponents.

$$S(k,\omega) \simeq \mathscr{A}(k) \cdot \xi(\omega - \varepsilon_h(k)) \cdot [\omega - \varepsilon_h(k)]^{\vartheta}$$

- ⋆ **'09** (Affleck, Pereira, White) X-ray edge-type model $\rightsquigarrow$ predictions for edge exponents.
- ⋆ **'10** (Caux, Glazman, Imambekov, Shashi) Predictions for $\mathscr{A}(k)$ (NLSM);
- Can these predictions be confirmed by a computation from the microscopic model?

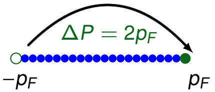
Long-distance asymptotics of densities at $T = 0K$

'11 Kitanine, Kozlowski, M., Slavnov, Terras

spin-spin correlation function of the XXZ chain at $T = 0K$:

$$\frac{\left\langle G.S. \middle| \sigma_1^z \sigma_m^z \middle| G.S. \right\rangle}{\left\langle G.S. \middle| G.S. \right\rangle} = \langle \sigma^z \rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2}\left(1 + \mathrm{o}(1)\right) + \sum_{\ell=1}^{+\infty} \frac{2\cos\left(2m\ell p_F\right)}{m^{2\ell^2\mathcal{Z}^2}} \cdot |\mathcal{F}_\ell|^2 \left(1 + \mathrm{o}(1)\right)$$

$$|\mathcal{F}_\ell|^2 = \lim_{L \to +\infty} \left(\frac{L}{2\pi}\right)^{2\ell^2\mathcal{Z}^2} \frac{\left|\left\langle G.S. \middle| \sigma_1^z \middle| \mathrm{umkp} - \ell \right\rangle\right|^2}{\left\|G.S.\right\|^2 \cdot \left\|\mathrm{umkp} - \ell\right\|^2}$$



★ ground state in positive chemical potential

★ one Umklapp excitation $\Delta E = 0$ $\Delta P = 2p_F$.

✔ Confirms CFT and Luttinger liquid predictions.

✔ Agrees with RHP approach (**'08** KKMST).

✔ Similar results for NLSM.

T=0K leading harmonics in long-time & distance asymptotics

'12 Kitanine, Kozlowski, M., Slavnov, Terras

Currents: $j(x,t) \equiv e^{iHt}\Psi^{\dagger}(x)\Psi(x)e^{-iHt}$ asymptotic regime $x \to +\infty$ and x/t fixed.

Overall structure of the asymptotic series (space-like regime) :

$$\left\langle j(x,t)\, j(0,0)\right\rangle = \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\left(x^2-t^2v_F^2\right)^2}\left(1+\mathrm{o}(1)\right)$$

$$+\sum_{\substack{\ell_+;\ell_-\in\mathbb{Z} \\ \ell_++\ell_-\leq 0}}^{*} \frac{e^{ix\ell_+p_F}}{\left[-i(x-v_Ft)\right]^{\Delta^{(R)}_{\ell_+;\ell_-}}}\,\frac{e^{-ix\ell_-p_F}}{\left[i(x+v_Ft)\right]^{\Delta^{(L)}_{\ell_+;\ell_-}}}$$

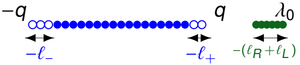
$$\times\, e^{-i(\ell_++\ell_-)[xp(\lambda_0)-t\varepsilon(\lambda_0)]}\left(\frac{[p'(\lambda_0)]^2}{-i[xp''(\lambda_0)-t\varepsilon''(\lambda_0)]}\right)^{\frac{|\ell_++\ell_-|^2}{2}}\cdot\frac{(2\pi)^{\frac{|\ell_++\ell_-|}{2}}\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2}{G\left(1+|\ell_++\ell_-|\right)}\left(1+\mathrm{o}(1)\right).$$

- ★ λ_0 **Saddle-point** of the oscillating phase: $p'(\lambda_0) - t\varepsilon'(\lambda_0)/x = 0$.
  - ⇝ p dressed momentum & ε dressed energy.

Form factor interpretation of the amplitudes

$$\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{|\ell_++\ell_-|^2+\Delta^{(R)}_{\ell_+;\ell_-}+\Delta^{(L)}_{\ell_+;\ell_-}} \cdot \frac{\left|\langle G.S.| j(0) |\mathrm{Ex}(\ell_+;\ell_-)\rangle\right|^2}{\| G.S.\|^2 \cdot \|\mathrm{Ex}(\ell_+;\ell_-)\|^2}\right\}$$

★ ℓ_+: # additional particles at q ℓ_-: # additional particles at $-q$ $|\ell_+ + \ell_-|$: # particles at λ_0



★ ground state in positive chemical potential

★ excitation $\begin{cases} \Delta E = |\ell_+ + \ell_-|\varepsilon(\lambda_0) \\ \Delta P = |\ell_+ + \ell_-| p(\lambda_0) - |\ell_+| p_F - |\ell_-|(-p_F) \end{cases}$

- Critical exponents $\Delta^{(R/L)}_{\ell_+;\ell_-}$ originate from excitations on Fermi boundaries.

$$\Delta^{(R)}_{\ell_+;\ell_-} = (\ell_+ + \ell_-)\phi(q,\lambda_0) - \ell_-\phi(q,-q) - \ell_+\phi(q,q) \qquad \left(I - \frac{K}{2\pi}\right)\cdot \phi(\lambda,\mu) = \theta(\lambda-\mu)$$

- Critical exponent $\frac{|\ell_+ + \ell_-|^2}{2}$ originates from gaussian saddle-point.

✔ Agrees with the first terms obtained through Natte series (**'11** Kozlowski, Terras).

The power-law behavior of dynamical structure factors (NLSM)

'12 Kitanine, Kozlowski, M., Slavnov, Terras

(k,ω) configuartion close to the hole excitation line

$$(p_F - p(\lambda_0)\,,-\varepsilon(\lambda_0)) \qquad \text{with} \qquad \lambda_0 \in \,]-q\,;q[\;.$$

★ The *hole* treshold

$$S\Big(p_F - p(\lambda_0), -\varepsilon(\lambda_0) + \delta\omega\Big) \simeq \frac{\xi\,(\delta\omega)\,[\delta\omega]^{\Delta^{(R)}_{1;0} + \Delta^{(L)}_{1;0} - 1}}{[v+v_F]^{\Delta^{(R)}_{1;0}}\,[v_F - v]^{\Delta^{(L)}_{1;0}}} \cdot \frac{(2\pi)^2\,\big|\mathcal{F}^{(j)}_{1,0}\big|^2}{\Gamma(\Delta^{(R)}_{1;0} + \Delta^{(L)}_{1;0})}\;.$$

★ v: velocity of the hole at λ_0 $\qquad$ v_F: velocity excitations on Fermi boundary.

$$\big|\mathcal{F}^{(j)}_{1,0}\big|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{1+\Delta^{(R)}_{1;0} + \Delta^{(L)}_{1;0}} \frac{\Big|\big\langle G.S.\big|\,j\,(0)\,\big|\mathrm{Ex}\big\rangle\Big|^2}{\big\|G.S.\big\|^2\cdot\big\|\mathrm{Ex}\big\|^2}\right\}$$

★ ground state

★ excitation $\begin{cases} \Delta E &= -\varepsilon\,(\lambda_0) \\ \Delta P &= p_F - p(\lambda_0) \end{cases}$

(k,ω) configuration close to the particle excitation line

$$\big(p(\lambda_0)-p_F,\varepsilon(\lambda_0)\big) \qquad \text{with} \qquad \lambda_0 \in \,]\, q\,;+\infty\,[\ .$$

⋆ The *particle* treshold

$$S\Big(p(\lambda_0)-p_F,\varepsilon(\lambda_0)+\delta\omega\Big) \simeq \frac{[\delta\omega]^{\Delta_{-1;0}^{(R)}+\Delta_{-1;0}^{(L)}-1}}{[v+v_F]^{\Delta_{-1;0}^{(R)}}[v_F-v]^{\Delta_{-1;0}^{(L)}}}\cdot\frac{(2\pi)^2\,\big|\mathcal{F}_{-1,0}^{(j)}\big|^2}{\Gamma\big(\Delta_{1;0}^{(R)}+\Delta_{1;0}^{(L)}\big)}$$
$$\times\ \frac{\xi(\delta\omega)\sin\big[\pi\Delta_{-1;0}^{(L)}\big]+\xi(-\delta\omega)\sin\big[\pi\Delta_{-1;0}^{(R)}\big]}{\sin\pi\big[\Delta_{-1;0}^{(R)}+\Delta_{-1;0}^{(L)}\big]}$$

✔ Microscopic model approach ⇝ the non-linear Luttinger-based predictions.

The form factor approach

Form factor expansion for finite L of $\mathcal{O}(x,t) \equiv e^{iHt}\mathcal{O}(x)\, e^{-iHt}$

$$\langle G.S.|\mathcal{O}(x,t)\,\mathcal{O}^{\dagger}(0,0)\,|G.S.\rangle = \sum_{\{\mu\}_{\text{ex}}} \langle G.S.|e^{-ixP+itH}\mathcal{O}(0,0)\,e^{ixP-itH}|\,\{\mu\}_{\text{ex}}\rangle\langle\{\mu\}_{\text{ex}}\,|\mathcal{O}^{\dagger}(0,0)\,|G.S.\rangle$$

$$= \sum_{\{\mu\}_{\text{ex}}} e^{ix(P_{\text{G.S.}}-P_{\text{ex}})-it(\mathcal{E}_{G.S.}-\mathcal{E}_{\text{ex}})}\left|\langle G.S.|\mathcal{O}(0,0)\,|\,\{\mu\}_{\text{ex}}\rangle\right|^{2}$$

Steps of the computation

- Characterize the excitations above the ground state;
- Asymptotic in *size* L formula for $\langle G.S.|\mathcal{O}(0,0)|\,\{\mu\}_{\text{ex}}\rangle$;
- Localize sums at stationary-points: saddle-point, ends of Fermi zone ;
- Sum-up in the asymptotic regime.

Free fermion model in finite volume

- Eigenfunctions $\rightsquigarrow$ from plane-waves $\varphi(\boldsymbol{x} \mid \{\lambda_a\}_1^N) = \exp\left\{i \sum_{k=1}^{N} \lambda_k x_k\right\}$
- Boundary conditions λ_a $\rightsquigarrow$ quantization of momenta $\lambda_a = \frac{2\pi}{L} n_a$ for some integers n_a .
- Simple form of spectrum $\mathcal{E}(\{\lambda_a\}_1^N) = \sum_{a=1}^{N} \lambda_a^2$ and $\mathcal{P}(\{\lambda_a\}_1^N) = \sum_{a=1}^{N} \lambda_a$

Ground state Momenta tightly packed around origin $\rightsquigarrow$ $n_a = a - (N+1)/2$
Particle-hole excitations $\rightsquigarrow$ other choices of integers:

$$n_j = j - \frac{N+1}{2} \quad \text{for } j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\} \qquad \text{and} \qquad n_{h_a} = p_a - \frac{N+1}{2} \quad \text{for } a \in \{1, \dots, n\}$$

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \dots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \dots, \mu_{p_n}$

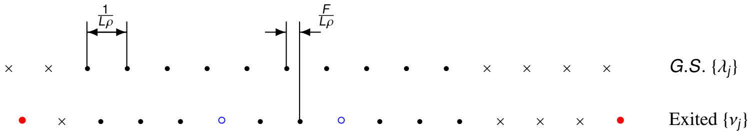
$\Rightarrow$ Excitation spectrum is additive.

$$\mathcal{P}_{\text{ex}} - \mathcal{P}_{G.S.} = \sum_{a=1}^{n} \mu_{p_a} - \mu_{h_a} \quad \text{and} \quad \mathcal{E}_{\text{ex}} - \mathcal{E}_{G.S.} = \sum_{a=1}^{n} \mu_{p_a}^2 - \mu_{h_a}^2$$

Excited states in the interacting case

Particle-hole excitations

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \dots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \dots, \mu_{p_n}$



$\Rightarrow$ Excited state's rapidities ν_j shifted infinitesimally in respect to GS rapidities λ_j.

$$\nu_j - \lambda_j = \frac{1}{L\rho(\lambda_j)} \cdot F\left(\lambda_j \,\middle|\, \begin{matrix} \mu_{p_1}, \dots, \mu_{p_n} \\ \mu_{h_1}, \dots, \mu_{h_n} \end{matrix}\right) + \mathrm{O}(L^{-2}) \qquad j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\} \;.$$

$\Rightarrow$ Additive excitation spectrum.

$$\mathcal{P}_{\text{ex}} - \mathcal{P}_{G.S.} = \sum_{a=1}^{n} p(\mu_{p_a}) - p(\mu_{h_a}) + \mathrm{O}(L^{-1}) \quad \text{and} \quad \mathcal{E}_{\text{ex}} - \mathcal{E}_{G.S.} = \sum_{a=1}^{n} \varepsilon(\mu_{p_a}) - \varepsilon(\mu_{h_a}) + \mathrm{O}(L^{-1})$$

Excitations on the Fermi boundaries

⊛ n-particle hole excitations with macroscopic momenta $\{\mu_{p_a}\}_1^n$, $\{\mu_{h_a}\}_1^n$ *on* the Fermi surface

- n_h^+ holes and n_p^+ particles on right Fermi zone $\Rightarrow$ local deficiency $\ell \equiv n_p^+ - n_h^+$;
- n_h^- holes and n_p^- particles on left Fermi zone $\Rightarrow$ local deficiency $-\ell \equiv n_p^- - n_h^-$.

⇝ parametrization in terms of effective integers $h_a^\pm$ and $p_a^\pm$

$$\begin{aligned} \mu_{p_a} &\sim q + \frac{2\pi}{L\rho(q)} p_a^+ \qquad \text{or} \qquad \mu_{p_a} \sim -q - \frac{2\pi}{L\rho(q)} p_a^- \\ \mu_{h_a} &\sim q - \frac{2\pi}{L\rho(q)} h_a^+ \qquad \text{or} \qquad \mu_{h_a} \sim -q + \frac{2\pi}{L\rho(q)} h_a^- \end{aligned}$$

- Simple form for the excitation momentum

$$\mathcal{P}_{\text{ex}} - \mathcal{P}_{\text{G.S.}} \sim 2\ell p_F + \frac{2\pi}{L}\left(\sum_{a=1}^{n_p^+} p_a^+ + \sum_{a=1}^{n_h^+} h_a^+ \right) - \frac{2\pi}{L}\left(\sum_{a=1}^{n_p^-} p_a^- + \sum_{a=1}^{n_h^-} h_a^- \right).$$

Asymptotic behavior of form factors: the result

NLSE, **'90** Slavnov , *XX* **'06** Arikawa,Karbach,Müller,Wiele
6-Vertex R matrix **'09-'10** Kitanine, Kozlowski, M., Slavnov, Terras

- excited state with particles $\mu_{p_1}, \ldots, \mu_{p_n}$ and holes $\mu_{h_1}, \ldots, \mu_{h_n}$.
- F shift function associated to such excitation.
- $\{\lambda_a\}_1^N$ GS distr. momenta, $\{\nu_a\}_1^{N'}$ excited state momenta.

Structural assumption

Fermi repulsion-like behavior of the form factor (XXZ exact results : **'99** Kitanine, M., Terras)

$$\frac{\langle\, Excited\, | \mathcal{O}\,(0,0)\,|\, G.S.\,\rangle}{\|Excited\| \cdot \|G.S.\|} \sim \frac{\prod_{j<k}^{N} (\lambda_j - \lambda_k) \prod_{j>k}^{N'} (\nu_j - \nu_k)}{\prod_{k=1}^{N} \prod_{j=1}^{N'} (\lambda_k - \nu_j)} \times \underbrace{\mathscr{A}\begin{pmatrix} \mu_{p_1}, \ldots, \mu_{p_n} \\ \mu_{h_1}, \ldots, \mu_{h_n} \end{pmatrix}}_{\text{regular}} .$$

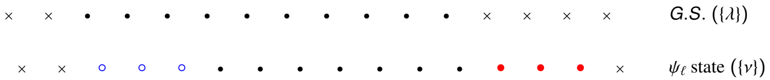
⊛ Extract the large volume L behavior $\Longrightarrow$ many cancellation of terms going to zero with L.

The power-law decay of form factors

- Algebraic decay of form factors (in the volume L)

$$\left| \frac{\langle \, Excited \, | \mathcal{O}\,(0,0) \,|\, G.S. \, \rangle}{\|Excited\| \cdot \|G.S.\|} \right|^2 \sim \left(\frac{2\pi}{L} \right)^{\theta[F]} \cdot \underbrace{\mathcal{R}_n \begin{pmatrix} \{p_a\};\{\mu_{p_a}\} \\ \{h_a\};\{\mu_{h_a}\} \end{pmatrix} [F]}_{\text{discrete}} \cdot \underbrace{\mathcal{A}_n \begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix}}_{\text{smooth}} .$$

- Excitation on the Fermi boundary $\Longrightarrow$ description in terms of ℓ-shifted states



- Local shifts of rapidities $N, L \gg s$:

$$\nu_{N-s} - \lambda_{N-s} \sim s \cdot \frac{F_{\ell;+}}{L\rho(q)} \quad \text{right Fermi} \quad \text{and} \quad \nu_s - \lambda_s \sim s \cdot \frac{F_{\ell;-}}{L\rho(-q)} \quad \text{left Fermi}$$

- one value for volume power θ_ℓ.

Form factors of ℓ-shifted states

$$\left|\mathcal{F}_\ell\right|^2 = \lim_{L\to+\infty}\left\{ L^{\theta_\ell}\left|\frac{\langle\, G.S.\,|\mathcal{O}|\,\psi_\ell\,\rangle}{\|G.S\|\cdot\|\psi_\ell\|}\right|^2\right\} \qquad \text{model/operator dependent} .$$

⊛ Form factors of any low-lying excitation with ℓ particles more on *right* Fermi zone:

$$\left|\frac{\langle\, Ex\,|\mathcal{O}\,(0,0)\,|\,G.S.\,\rangle}{\|Ex\|\,\cdot\,\|G.S.\|}\right|^2 \sim \frac{\left|\mathcal{F}_\ell\right|^2}{L^{\theta_\ell}} \times \frac{G^2(1+F_{\ell;+})G^2(1-F_{\ell;-})}{G^2(1+\ell+F_{\ell;+})G^2(1-\ell-F_{\ell;-})}\left(\frac{\sin(\pi F_{\ell;+})}{\pi}\right)^{2n_h^+}$$
$$\times\left(\frac{\sin(\pi F_{\ell;-})}{\pi}\right)^{2n_h^-} R_{n_p^+,n_h^+}\Big(\{p_a^+\},\{h_a^+\}\mid F_{\ell;+}\Big) R_{n_p^-,n_h^-}\Big(\{p_a^-\},\{h_a^-\}\mid -F_{\ell;-}\Big) .$$

⊛ **Red** part is universal. $\quad G \rightsquigarrow$ Barnes function.

$$R_{n,m}\Big(\{p_a\}_1^n,\{h_a\}_1^m \mid F\Big) \equiv \frac{\prod_{j>k}^{n}(p_j-p_k)^2\prod_{j>k}^{m}(h_j-h_k)^2}{\prod_{j=1}^{n}\prod_{k=1}^{m}(p_j+h_k-1)^2}\prod_{k=1}^{n}\frac{\Gamma^2(p_k+F)}{\Gamma^2(p_k)}\prod_{k=1}^{m}\frac{\Gamma^2(h_k-F)}{\Gamma^2(h_k)} .$$

Form factor expansion of the generating function

$$\left\langle \mathcal{O}(x)\mathcal{O}^{\dagger}(0)\right\rangle = \sum_{\{\nu\}_{\mathrm{ex}}} \mathrm{e}^{ix(P_{\mathrm{G.S.}}-P_{\mathrm{ex}})} \left|\langle G.S.|\mathcal{O}(0,0)|\,\{\nu\}_{\mathrm{ex}}\,\rangle\right|^2$$

The $x \to +\infty$ asymptotics

- Only states having the same per-site energy as GS contribute in $L \to +\infty$;
- Only the individual leading in L behavior contributes to $L \to +\infty$ limit;

$$\left|\langle G.S.|\mathcal{O}(0,0)|\,\{\nu\}_{\mathrm{ex}}\,\rangle\right|^2 \sim L^{-\theta[\{\mu\}_{\mathrm{ex}}]}\,\mathcal{F}(\{\mu\}_{\mathrm{ex}})$$

$$\mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{\mathrm{G.S.}} = \sum_{a=1}^{n} p(\mu_{p_a}) - p(\mu_{h_a}) + \mathrm{O}\left(L^{-1}\right) \qquad \mathcal{E}_{\mathrm{ex}} - \mathcal{E}_{\mathrm{G.S.}} = \sum_{a=1}^{n} \varepsilon(\mu_{p_a}) - \varepsilon(\mu_{h_a}) + \mathrm{O}\left(L^{-1}\right)$$

- Approximate summand at stationary points $\rightsquigarrow$ endpoints of Fermi zone ;
- sum-up the resulting *critical* series .

The effective form factor series I

$$\langle \mathcal{O}(x)\mathcal{O}^{\dagger}(0)\rangle = \sum_{n=0}^{N} \sum_{p_1<\cdots<p_n} \sum_{h_1<\cdots<h_n} \left(\frac{2\pi}{L}\right)^{\theta[F]} \prod_{a=1}^{n} \left\{ \frac{e^{ixp(\mu_{p_a})}}{e^{ixp(\mu_{h_a})}} \right\} \cdot \mathcal{R}_n\begin{pmatrix} \{p_a\};\{\mu_{p_a}\} \\ \{h_a\};\{\mu_{h_a}\} \end{pmatrix}[F] \cdot \mathcal{A}_n\begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix}$$

- Smooth part and state depending shift function.
- Stationary points
  - Endpoints of the Fermi zone $\begin{cases} \text{holes} \in \{1,\dots,N\} & \rightsquigarrow \quad \mu_h \in [-q;q] \\ \text{particles} \in \mathbb{Z}\setminus\{1,\dots,N\} & \rightsquigarrow \quad \mu_p \in \mathbb{R}\setminus[-q;q] \end{cases}$

Partition the domain according to the stationary points and keep **only** leading contributions

- Stationary points of the space-like regime:
  - Particle/hole excitations on `right` Fermi boundary and ℓ *additional* particles ;
  - Particle/hole excitations on `left` Fermi boundary and $-\ell$ *additional* particles .
- Partition sums according to `right`, `left` Fermi zones

$$\{p_a\}_1^n = \{N + p_a^+\}_1^{n_p^+} \cup \{1 - p_a^-\}_1^{n_p^-} \qquad \text{and} \qquad \{h_a\}_1^n = \{N + 1 - h_a^+\}_1^{n_h^+} \cup \{h_a^-\}_1^{n_h^-} .$$

There are *particle* deficiencies on Fermi boundaries : $n_h^+ = n_p^+ - \ell$ and $n_h^- = n_p^- + \ell$.

- Keep leading approximation of phases and form factors.

Several algebraic manipulations later ...

The form of the series at $x \to +\infty$

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \lim_{N,L\to+\infty} \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{i2x\ell p_F} \cdot \left|\mathcal{F}_{\ell}\right|^2 \cdot \mathscr{R}_{\ell}\big(x \mid F_{\ell;+}\big)\mathscr{R}_{-\ell}\big(-x \mid -F_{\ell;-}\big)$$

$$\mathscr{R}_{\ell}(x \mid \nu) = \left(\frac{2\pi}{L}\right)^{(\nu+\ell)^2} \frac{G^2(1+\nu)}{G^2(1+\nu+\ell)} \sum_{\substack{n_p,n_h\geq 0\\ n_p-n_h=\ell}} \sum_{\substack{p_1<\dots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\dots<h_{n_h}\\ h_a\in\mathbb{N}^*}} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h} \prod_{a=1}^{n_p}\left\{\mathrm{e}^{\frac{2i\pi}{L}p_a x}\right\} \cdot \prod_{a=1}^{n_h}\left\{\mathrm{e}^{\frac{2i\pi}{L}(h_a-1)x}\right\}$$

$$\frac{\prod\limits_{a<b}^{n_p}(p_a-p_b)^2 \cdot \prod\limits_{a<b}^{n_h}(h_a-h_b)^2}{\prod\limits_{a=1}^{n_p}\prod\limits_{b=1}^{n_h}(p_a+h_b-1)^2} \cdot \prod_{a=1}^{n_p}\Gamma^2\begin{pmatrix} p_a+\nu \\ p_a \end{pmatrix} \prod_{a=1}^{n_h}\Gamma^2\begin{pmatrix} h_a-\nu \\ h_a \end{pmatrix}$$

The form of the series at $x \to +\infty$

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \sim \lim_{N,L\to+\infty} \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{i2x\ell p_F} \cdot \left|\mathcal{F}_\ell\right|^2 \cdot \mathscr{R}_\ell\big(x \mid F_{\ell;+}\big)\mathscr{R}_{-\ell}\big(-x \mid -F_{\ell;-}\big)$$

$$\mathscr{R}_\ell(x \mid \nu) = \left(\frac{2\pi}{L}\right)^{(\nu+\ell)^2} \frac{G^2(1+\nu)}{G^2(1+\nu+\ell)} \sum_{\substack{n_p,n_h\geq 0\\ n_p-n_h=\ell}} \sum_{\substack{p_1<\dots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\dots<h_{n_h}\\ h_a\in\mathbb{N}^*}} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h} \prod_{a=1}^{n_p}\left\{\mathrm{e}^{\frac{2i\pi}{L}p_a x}\right\} \cdot \prod_{a=1}^{n_h}\left\{\mathrm{e}^{\frac{2i\pi}{L}(h_a-1)x}\right\}$$

$$\frac{\prod\limits_{a<b}^{n_p}(p_a-p_b)^2 \cdot \prod\limits_{a<b}^{n_h}(h_a-h_b)^2}{\prod\limits_{a=1}^{n_p}\prod\limits_{b=1}^{n_h}(p_a+h_b-1)^2} \cdot \prod_{a=1}^{n_p}\Gamma^2\left(\begin{array}{c} p_a+\nu \\ p_a \end{array}\right) \prod_{a=1}^{n_h}\Gamma^2\left(\begin{array}{c} h_a-\nu \\ h_a \end{array}\right)$$

$$\mathscr{R}_\ell(x \mid \nu) = \left(\frac{2\pi/L}{1-\mathrm{e}^{\frac{2i\pi}{L}x}}\right)^{(\nu+\ell)^2}$$

- $\ell = 0$ Z-measures on partitions (**Kerov-Vershik,Borodin-Olshanski, Okounkov**) ;
- generalization to $\ell \neq 0$ and alternative proof at $\ell = 0$ (**'11, KKMST**).

The last step

$$\left\langle \mathcal{O}\left(x\right)\mathcal{O}^{\dagger}\left(0\right)\right\rangle \sim \lim_{N,L\to+\infty}\sum_{\ell\in\mathbb{Z}} \mathrm{e}^{i2x\ell p_F}\cdot\left|\mathcal{F}_{\ell}\right|^{2}\cdot\left(\frac{2\pi/L}{1-\mathrm{e}^{\frac{2i\pi}{L}x}}\right)^{(F_{\ell;+}+\ell)^2}\left(\frac{2\pi/L}{1-\mathrm{e}^{-\frac{2i\pi}{L}x}}\right)^{(F_{\ell;-}+\ell)^2} .$$

Now easy to send $L\to+\infty$

$$\left\langle \mathcal{O}\left(x\right)\mathcal{O}^{\dagger}\left(0\right)\right\rangle \sim \sum_{\ell\in\mathbb{Z}} \frac{\mathrm{e}^{i2x\ell p_F}\cdot\left|\mathcal{F}_{\ell}\right|^{2}}{(-ix)^{\Delta_{\ell;+}}\cdot(ix)^{\Delta_{\ell;-}}} .$$

Structure of the asymptotics

- Asymptotics indexed by Umklapp excitations ℓ ;
- $\left|\mathcal{F}_{\ell}\right|^{2}$ model dependent **but** universal interpretation ;
- Critical exponent $\Delta_{\ell;+}=(F_{\ell;+}+\ell)^2$ and $\Delta_{\ell;-}=(F_{\ell;-}+\ell)^2$.

The XXZ results

$\rightsquigarrow$ leading asymptotic terms for $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2 \sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \, \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

with $|\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2\mathcal{Z}^2} \frac{|\langle \psi_g | \sigma_1^z | \psi_\ell \rangle|^2}{\|\psi_g\|^2 \, \|\psi_\ell\|^2}$,

$|\psi_\ell\rangle$ being the ℓ-shifted ground state.

$\rightsquigarrow$ leading asymptotic terms for $\langle \sigma_1^+ \, \sigma_{m+1}^- \rangle$:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \, |\mathcal{F}_\ell^+|^2_{\text{finite}} \, \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

$$\left|\mathcal{F}_\ell^+\right|^2_{\text{finite}} = \lim_{M\to\infty} M^{(2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2})} \frac{|\langle \psi_g | \sigma_1^+ | \psi_\ell \rangle|^2}{\|\psi_g\|^2 \, \|\psi_\ell\|^2}$$

$|\psi_\ell\rangle$ being the ℓ-shifted ground state in the (N_0+1)-sector.

with $\mathcal{Z} = Z(\pm q)$ and $Z(\lambda) + \frac{1}{2\pi}\int_{-q}^{q} d\mu \, \frac{\sin 2\zeta}{\sinh(\lambda-\mu+i\zeta)\sinh(\lambda-\mu-i\zeta)} \, Z(\mu) = 1$. At free fermion point, $\zeta = \pi/2$, and $\mathcal{Z} = 1$ for zero magnetic field.

The n-point correlation functions

'13, to appear Kitanine, Kozlowski, M., Terras

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) \;=\; \langle\,\Psi_g\,|\mathcal{O}_1(x_1)\ldots\mathcal{O}_r(x_r)|\,\Psi_g\,\rangle\,,$$

Local operators $\mathcal{O}_a(x)$ connect states with N and $N+o_a$ pseudo-particles; the form factor expansion given as a multiple sum over intermediate normalized states $|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle$ with $s=1,\ldots r-1$, labelled by sets of integers corresponding to particles and holes excitations :

$$\mathcal{I}_n^{(s)} \;=\; \Big\{\{p_a^{(s)}\}_1^n \quad ; \quad \{h_a^{(s)}\}_1^n\Big\}$$

$$\langle\,\Psi\big(\mathcal{I}_m^{(s-1)}\big)\,|\mathcal{O}_s(x)|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle \;=\; \mathrm{e}^{ix(\Delta\mathcal{P})_{s-1}^{s}}\cdot\mathcal{F}_{\mathcal{O}_s}\Big(\mathcal{I}_m^{(s-1)}\;|\;\mathcal{I}_n^{(s)}\Big)$$

$$(\Delta\mathcal{P})_{s-1}^{s} \;=\; \mathcal{P}_{\mathcal{I}_m^{(s-1)}} - \mathcal{P}_{\mathcal{I}_n^{(s)}}$$

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) \;=\; \prod_{s=1}^{r-1}\Big\{\sum_{\{\mathcal{I}_{n(s)}^{(s)}\}}\Big\}\cdot\prod_{s=1}^{r-1}\Big\{\exp\Big[i(x_{s+1}-x_s)\cdot\Delta\mathcal{P}(\mathcal{I}_{n(s)}^{(s)})\Big]\Big\}\cdot\prod_{s=1}^{r}\mathcal{F}_{\mathcal{O}_s}\Big(\mathcal{I}_{n(s-1)}^{(s-1)}\;|\;\mathcal{I}_{n(s)}^{(s)}\Big)$$

General form factors (1)

$$\mathcal{F}_{O_s}\left(\mathcal{I}_m^{(s-1)}\middle|\mathcal{I}_n^{(s)}\right) = \mathcal{F}_{O_s}(\ell_{s-1},\ell_s)\cdot C^{(\ell_{s-1};\ell_s)}\left(\nu_s^+,\nu_s^-\right)\times \mathscr{F}^{(+)}\left[\mathcal{J}^{(s-1)}_{m_{p;+};m_{h;+}};\mathcal{J}^{(s)}_{n_{p;+};n_{h;+}}\mid \nu_s^+\right]\cdot$$

$$\cdot\mathscr{F}^{(-)}\left[\mathcal{J}^{(s-1)}_{m_{p;-};m_{h;-}};\mathcal{J}^{(s)}_{n_{p;-};n_{h;-}}\mid \nu_s^-\right]$$

$$\mathscr{F}_{O_s}(\ell_{s-1},\ell_s) = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{\rho_s(\nu_s^+)+\rho_s(\nu_s^-)}\langle\,\Psi\left(\mathcal{L}^{(s-1)}_{\ell_{s-1}}\right)|O_s(0)|\,\Psi\left(\mathcal{L}^{(s)}_{\ell_s}\right)\rangle\right\}$$

$$\rho_s(\nu) = \frac{1}{2}(\ell_s-\ell_{s-1})^2 + \frac{1}{2}\nu^2 - (\ell_s-\ell_{s-1})\nu\,.$$

$$\nu_s^+ = \nu_s(q) - o_s \quad \text{and} \quad \nu_s^- = \nu_s(-q)$$

in terms of the relative shift function between the ℓ_s,ℓ_{s-1} critical states

$$\nu_s(\lambda) = F_{s-1}(\lambda) - F_s(\lambda)\,.$$

General form factors (2)

The right Fermi boundary critical form factor reads :

$$\mathscr{F}^{(+)}\left[\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t} \mid \nu\right] = \left(\frac{2\pi}{L}\right)^{\rho_s(\nu)} (-1)^{n_t} \left(\frac{\sin[\pi\nu]}{\pi}\right)^{n_t+n_h} \varpi\left(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t} \mid \nu\right) \cdot$$

$$\frac{\prod_{a<b}^{n_p}(p_a-p_b)\prod_{a<b}^{n_h}(h_a-h_b)}{\prod_{a=1}^{n_p}\prod_{b=1}^{n_h}(p_a+h_b-1)} \frac{\prod_{a<b}^{n_k}(k_a-k_b)\prod_{a<b}^{n_t}(t_a-t_b)}{\prod_{a=1}^{n_k}\prod_{b=1}^{n_t}(k_a+t_b-1)} \Gamma\left(\begin{matrix}\{p_a+\nu\} & \{h_a-\nu\} & \{k_a-\nu\} & \{t_a+\nu\} \\ \{p_a\} & \{h_a\} & \{k_a\} & \{t_a\}\end{matrix}\right)$$

$$\varpi\left(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t} \mid \nu\right) = \prod_{a=1}^{n_h}\left\{\frac{\prod_{b=1}^{n_k}\left(1-k_b-h_a+\nu\right)}{\prod_{b=1}^{n_t}\left(t_b-h_a+\nu\right)}\right\}\cdot\prod_{a=1}^{n_p}\left\{\frac{\prod_{b=1}^{n_t}\left(p_a+t_b+\nu-1\right)}{\prod_{b=1}^{n_k}\left(p_a-k_b+\nu\right)}\right\}.$$

This ϖ term couples the right and left states particles and holes integers (not present if one of them is the ground state) hence leading to coupling of previous combinatorial sums!

General sums (1)

$$C\Big(\boldsymbol{x}_r;\boldsymbol{o}_r\Big) \simeq \sum_{\substack{\boldsymbol{\ell}_{r-1}\\ \in\mathbb{Z}^{r-1}}} \Big(\frac{2\pi}{L}\Big)^{\vartheta(\boldsymbol{\ell}_{r-1},\boldsymbol{o}_r)} \prod_{s=1}^{r-1}\Big\{ e^{2i\ell_s(x_{s+1}-x_s)p_F}\Big\} \prod_{s=1}^{r}\Big\{ C^{(\ell_{s-1};\ell_s)}\Big(\nu_s^+,\nu_s^-\Big)\Big\}\,.$$

$$\prod_{s=1}^{r}\Big\{\mathcal{F}_{O_s}(\ell_{s-1},\ell_s)\Big\}\,\mathscr{S}^-_{\boldsymbol{\ell}_{r-1}}\Big(\Big\{\frac{2\pi}{L}(x_{s+1}-x_s)\Big\}_1^{r-1},\{\nu_s^-(\boldsymbol{\ell}_s)\}_1^r\Big)\,\mathscr{S}^+_{\boldsymbol{\ell}_{r-1}}\Big(\Big\{\frac{2\pi}{L}(x_{s+1}-x_s)\Big\}_1^{r-1},\Big\{\nu_s^+(\boldsymbol{\ell}_s)\Big\}_1^r\Big)$$

$$\vartheta(\boldsymbol{\ell}_{r-1},\boldsymbol{o}_r) = \frac{1}{2}\sum_{s=1}^{r}\Big\{(\nu_s^+)^2+(\nu_s^-)^2\Big\} - \sum_{s=1}^{r-1}\Big\{\Big(\nu_s^+ +\nu_s^- -\nu_{s+1}^+ -\nu_{s+1}^-\Big)\ell_s - 2\ell_s^2\Big\} - 2\sum_{s=2}^{r-1}\ell_s\ell_{s-1}$$

$$\mathscr{S}^{\pm}_{\boldsymbol{\ell}_{r-1}}(\{t_s\},\{\nu_s\}) = \prod_{s=1}^{r-1}\sum_{\substack{n_p^{(s)},n_h^{(s)}=0\\ n_p^{(s)}-n_h^{(s)}=\pm\ell_s}}^{+\infty}\ \sum_{\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}}\ \prod_{s=1}^{r-1}\mathcal{R}^{\pm}(\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}|\nu_s,\nu_{s+1};t_s)\prod_{s=2}^{r-1}\varpi\Big(\mathcal{J}^{(s-1)}_{n_p^{(s-1)};n_h^{(s-1)}};\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}}\mid\pm\nu_s\Big)$$

The summation in the above formula runs through all the possible choices of the sets of integers that parametrize the states

$$\mathcal{J}^{(s)}_{n_p^{(s)};n_h^{(s)}} = \Big\{\{p_a^{(s)}\}_1^{n_p^{(s)}}\ ;\ \{h_a^{(s)}\}_1^{n_h^{(s)}}\Big\}$$

General sums (2)

Amazingly, these generalized combinatorial sums can be computed exactly!

$$\mathscr{S}^{\pm}_{\boldsymbol{\ell}_{r-1}}\Big(\{t_s\}_1^{r-1},\{\nu_s\}_1^r\Big) = \prod_{s=1}^{r-1}\left\{ e^{\pm i t_s \frac{\ell_s(\ell_s+1)}{2}} G\begin{pmatrix} 1\pm(\ell_s-\nu_s), 1\pm(\ell_s+\nu_{s+1}) \\ 1\mp\nu_s, 1\pm\nu_{s+1}\end{pmatrix}\right\}$$

$$\times \prod_{s=2}^{r-1} G\begin{pmatrix} 1\pm\nu_s, 1\pm(\ell_{s-1}-\ell_s+\nu_s) \\ 1\mp(\ell_s-\nu_s), 1\pm(\ell_{s-1}+\nu_s)\end{pmatrix} \cdot \prod_{b>a}^{r}\Big(1-e^{\pm i\sum_{s=a}^{b-1} t_a}\Big)^{(\nu_a+\kappa_a)(\nu_b+\kappa_b)}$$

$$\kappa_s = \ell_{s-1}-\ell_s \quad \text{for} \quad s=1,\dots,r \qquad \text{so that} \qquad \sum_{a=1}^{r}\kappa_a = 0.$$

$$C\big(\boldsymbol{x}_r;\boldsymbol{o}_r\big) = \sum_{\substack{\boldsymbol{\kappa}_r\in\mathbb{Z}^r \\ \sum\kappa_a=0}} \prod_{s=1}^{r}\{e^{2ip_F\kappa_s x_s}\}\cdot\mathcal{F}\Big(\{\kappa_a\}_1^r;\{o_a\}_1^r\Big)\cdot$$

$$\prod_{s=1}^{r}\Big(\frac{2\pi}{L}\Big)^{\frac{1}{2}[\theta_s^+(\kappa_s)]^2+\frac{1}{2}[\theta_s^-(\kappa_s)]^2} \prod_{b>a}^{r}\left\{\Big[1-e^{\frac{2i\pi}{L}(x_b-x_a)}\Big]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)}\cdot\Big[1-e^{-\frac{2i\pi}{L}(x_b-x_a)}\Big]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)}\right\}$$

$$\theta_b^{\pm}(\kappa_b) = \nu_b^{\pm}+\kappa_b$$

Asymptotic behavior of n-point correlation functions

Taking the thermodynamic limit we arrive at the following n-point correlation function asymptotic behavior :

$$C\Big(\boldsymbol{x}_r;\boldsymbol{o}_r\Big) = \sum_{\substack{\boldsymbol{\kappa}_r\in\mathbb{Z}^r \\ \sum\kappa_a=0}} \prod_{s=1}^{r}\Big\{\mathrm{e}^{2ip_F\kappa_s x_s}\Big\}\cdot\mathcal{F}\Big(\{\kappa_a\}_1^r;\{o_a\}_1^r\Big)$$

$$\prod_{b>a}^{r}\Big\{\Big[\,\mathrm{i}(x_b-x_a)\Big]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)}\cdot\Big[-i(x_b-x_a)\Big]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)}\Big\}\,.$$

Note that the above asymptotic expansion provides one with an expression that is symmetric under a simultaneous permutation

$$\Big(\boldsymbol{x}_r,\boldsymbol{o}_r\Big)\mapsto\Big(\boldsymbol{x}_r^{\sigma},\boldsymbol{o}_r^{\sigma}\Big)\quad\text{with}\quad\boldsymbol{x}_r^{\sigma}=\Big(x_{\sigma(1)},\dots,x_{\sigma(r)}\Big)\qquad\sigma\in\mathfrak{S}_r\,.$$

This is directly related to locality, namely to the fact that the local operators $\mathcal{O}_r(x_r)$ commute at different distances and, in particular, in the long-distance regime.

Conclusion and perspectives

Results

- ✔ Leading asymptotics of **any** harmonic in long-distance
- ✔ **All** harmonics in long-distance and large-time for pure particle-hole spectrum
- ✔ Reproduction of edge exponents with amplitudes from ABA
- ✔ Leading asymptotic behavior of n-point correlation functions

What's next?

- ⊛ Include the effects of bound states (time dependent case)
- ⊛ Full test of CFT (OPE of local operators + structure constants)