Correlation functions of higher spin chains

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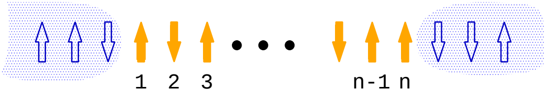
Based on Collaboration with
H. Boos, A. Klümper, F. Göhmann, D. Nawrath, A. Seel ...

Density Matrix Elements

A measure of correlations in finite segments of quantum spin chains.

- (reduced) Density Matrix Elements (DME)



$$D_n := \langle E_1 \otimes E_2 \cdots \otimes E_n \rangle$$

$$(D_n)^{\beta_1,\beta_2,\cdots,\beta_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n} := \langle E^{\alpha_1}_{\beta_1} E^{\alpha_2}_{\beta_2} \cdots E^{\alpha_n}_{\beta_n} \rangle \qquad (E^{\alpha}_{\beta})^i_j := \delta_{\alpha,i}\delta_{\beta,j}$$

- short correlation functions
- entanglement entropy

Main Problem Today

Problem

Consider the integrable isotropic $S = 1$ chain with L sites.

$$H = \frac{J}{4} \sum_{j=1}^{L} [\vec{S}_{j-1} \cdot \vec{S}_j - (\vec{S}_{j-1} \cdot \vec{S}_j)^2]$$

Evaluate reduced Density Matrix Elements D_n in $L \to \infty$ or its "inhomogenous" generalization $D_n(\xi_1, \cdots, \xi_n)$, at any T in a " factorized" form.

Outline of the talk

- A summary of DME results for $S = \frac{1}{2}$
  - multiple integral formulas
  - reduced $q-$ KZ , Hidden Grassmann structures
  - QTM, multiple integral formulas
- A description ofbulk thermodynamics of $S > \frac{1}{2}$
- Factorized DME of $S = 1, T > 0$
  - factorization via fusion.
  - factorization via difference equations

A review on $S = \frac{1}{2}$ at $T = 0$

- Multiple integral formula for $D_n(\xi_1, \cdots, \xi_n)$
  - Vertex Operator approach (Jimbo et al.(1992-))
  - $q-$KZ approach (Jimbo and Miwa (1994))
  - QISM : Solving inverse problems (Maillet et al (2000-)) valid even $h \neq 0$
- Factorize multiple integrals into sums of products of single integrals "by hand"
  - Boos Korepin Smirnov (2001-)
  - Sato Shiroishi Takahashi (2005) ($n = 8$)

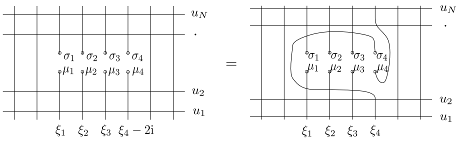
Conjecture (Boos-Korepin)

Correlation functions at $T = 0$ for $S = \frac{1}{2}$ XXX model are described by ($\ln 2$) and Riemann's ζ functions with odd arguments.

The reduced $q-$KZ equation and Hidden Grassmann

reduced $q-$KZ equation
(Boos et al (2004-))

- invariance of D_n under R
- reduction $D_n \to D_{n-1}$
- $D_n(\xi_1, \cdots, \xi_n - 2i) = AD_n(\xi_1, \cdots, \xi_n)$



Solution(Exponential formula)

- contains a transcendental fcn $\omega_{\alpha,q}$
- contains "Fermions"

It can explain

- factorized forms.
- finitely many terms in DME.
- only $\zeta(2k+1)$ appear

Status of DME $S = \frac{1}{2}, T > 0$

DME at $T > 0$ (Göhmann et al., JPA 36 (2005)) based on QTM

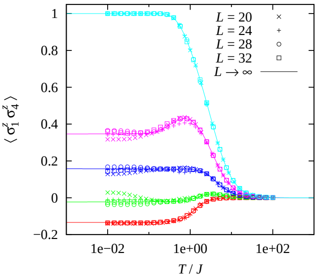
- Algebraic part : parallel to $T = 0$ case.
- Trotter limit : NLIE (Kluemper et al. (1991), Destri-de Vega , (1995))
- integrations contain "Fermi" (spinon) distribution functions $A(\lambda)$.

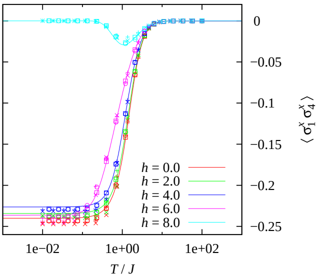
it can be explicitly factorized "by hand" (Wuppertal group (2006-))

- DME "=" "trigonometric" part + transcendental part
- only to replace $\omega_{\alpha,q}$ by its finite T analogue

factorization and Grassmann at $T > 0$

- Exponential formulas using "Fermions" are conjectured for $T > 0, h \neq 0$ (Wuppertal group (2006, 2007))
- partly proved ("Kyoto" group (2008))
- high precision calculation possible





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Evaluate Density Matrix Elements D_n in $L \to \infty$ or its "inhomogenous" generalization $D_n(\xi_1, \cdots, \xi_n)$, at any T s in a " factorized" form.

Aim of research

Common belief is..

- $S = \frac{1}{2}$ is fundamental.
- Description of $S > \frac{1}{2}$ is mere modification (at least for integrable cases)

What I believe is..

- Description of $S > \frac{1}{2}$ using $S = \frac{1}{2}$ is sometimes flawed.
- Each higher spins (composite particles) needs its own description of the Hilbert space.
- Natural description may offer an efficient formalism in numerics.

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Main problem Again

To be concrete , for $S = 1$

- multiple integral formula at $T = 0$ ✓
  - VO (Bougourzi et al., Konno)
  - QISM (Kitanine, Deguchi-Matsui)
- multiple integral formula at $T > 0$?
- factorization at $T > 0$?
- exponential formula at $T > 0$?

QTM

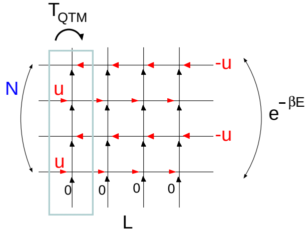
Main tool : QTM framework

M. Suzuki (1985), M. Suzuki and Inoue (1987), Koma(1987), J.S. et al (1990), Klümper (1992)

Map 1D quantum to 2D classical.

$$Z_{\rm 1Dquantum}(\beta, L) = Z_{\rm 2D}(N, L) = {\rm tr} T_{\rm QTM}(u)^L$$

$$u = -\frac{\beta}{N}$$



Theorem (M.Suzuki)

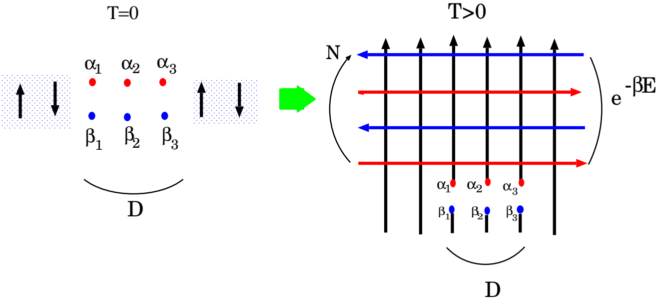
Only the largest eigenvalue of $T_{\rm QTM}$ contributes.

Neither summation nor variation necessary

DME in QTM formulation at $T > 0$

In QTM, no need to solve inverse problem (Göhmann et al., JPA 36)

$$\langle \mathrm{vac}|D|\mathrm{vac}\rangle \quad \Rightarrow \quad \frac{\mathrm{tr} D e^{-\beta H}}{Z}$$



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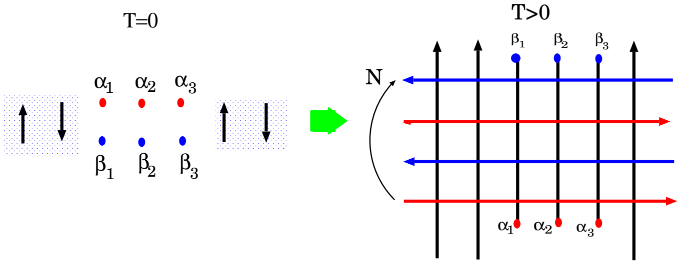


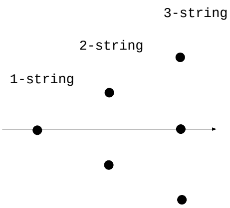
$$\left(D\right)^{\alpha_1,\cdots,\alpha_n}_{\beta_1,\cdots,\beta_n}(\xi_1,\cdots,\xi_n) = \frac{\langle\Psi|T^{\alpha_1}_{\beta_1}(\xi_1)\cdots T^{\alpha_n}_{\beta_n}(\xi_n)|\Psi\rangle}{\langle\Psi|T_{\rm QTM}(\xi_1)\cdots T_{\rm QTM}(\xi_n)|\Psi\rangle} \Rightarrow \text{parallel to } T=0!$$

BAE roots of $S > \frac{1}{2}$

Theorem (Gaudin, Tarasov)

Bethe ansatz roots characterizes highest weight states

composite states = strings
∞/∞=highly singular



Numerics on higher spins (Alcaraz et al '88)

- Ground state = 2S string
- Excited states= very complicated

Better not to deal with BAE roots directly.

Other descriptions?

Hilbert space of $S > \frac{1}{2}$

Conjecture(Reshetikhin '91)

Set $k = 2S$. The Hilbert space of higher spin chains will be decomposed into sums of products of Hilbert space of spinons and that of RSOS.
$\mathcal{H}_{\text{spin}S} = \mathcal{H}_{\text{spinon}} \oplus \mathcal{H}_{\text{RSOS}_k}$

- consistent with CFT limit

$\mathrm{SU}(2)_k$ WZW = Gaussian + Z_k parafermion

$$c = \frac{3k}{k+2} = 1 + \frac{2(k-1)}{k+2}$$

- Other confirmations
  - VO approach (Idzumi et al.)
  - CTM spectral decomposition (Arakawa et al.)

Auxiliary functions for $S > \frac{1}{2}$

Thermodynamics (JS '99):
consists of two pieces.

- "RSOS" pieces. $(1 \leq j \leq k-1)$
  $y_j, Y_j (:= 1 + y_j)$

=subset of Takahashi's TBA.

- Spinon pieces
  $\mathfrak{b}, \mathfrak{B}(:= 1 + \mathfrak{b})$

= a generalization of $\mathfrak{a}, \mathfrak{A}$

$\Downarrow$

finite number $(k+1)$ of objects!

They are nice objects, as

1. Good analyticity
2. They satisfy finitely many NLIE.
3. NLIE yields bulk quantities (specific heat..)
4. No need to deal directly with BAE roots.

NLIE for higher spins

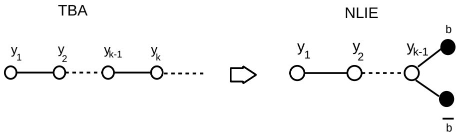
$T_j(x)$: fusion transfer matrix with auxiliary space = spin $\frac{j}{2}$

$$\Downarrow \quad y_j(x) = T_{j+1}(x)T_{j-1}(x)/\Phi_j(x) \quad \text{Klümper-Pearce('92)}$$

$$y_j(x+i)y_j(x-i) = \begin{cases} Y_{j+1}(x)Y_{j-1}(x) & 1 \le j \le k-2 \\ Y_{k-2}(x)\mathfrak{B}(x)\bar{\mathfrak{B}}(x) & j = k-1 \end{cases}$$

$$\mathfrak{b}(x) \sim \frac{Q(x+i(k+2))}{Q(x-ik)} T_{k-1}(x) \qquad \mathfrak{B}(x) \sim \frac{Q(x+ik)}{Q(x-ik)} T_k(x+i)$$

determine y_j in the physical strip $|\text{Im } x| \le 1$ and $\mathfrak{b}, 0 \le \text{Im } x < 1$.



multiple integral formula

Not dealing with BAE roots directly

- Good: no need to deal with singular objects
- No good: problem with DME

The algebraic part of calculation of DME goes parallel to $S = \frac{1}{2}$ case:

$$\langle T^{\alpha_1}_{\beta_1}(\xi_1) \cdots T^{\alpha_n}_{\beta_n}(\xi_n) \rangle \sim \sum_{\mathrm{BAEroots}\{\mu_j\} \cup \mathrm{others}} \mathcal{S}(\{\mu_j\})$$

- Zeros of Q (= BAE roots $\{\mu_j\}$) are not encoded in $\mathfrak{B}, Y_j!$, $\mathfrak{B}(\mu_j) \neq 0$
- No simple relation exists like,

$$\sum_{\mathrm{BAEroots}\{\mu_j\} \cup \mathrm{others}} \mathcal{S}(\{\mu_j\}) \sim \int \frac{d\lambda}{2\pi i \mathfrak{B}} S(\{\lambda\})$$

multiple integral formula II

Still we can

- adopt narrower contours separated in the upper and lower half planes
- impose "subtle relations" among these contours
- introduce one more auxiliary function $\mathfrak{f}, \mathfrak{F} := 1 + \mathfrak{f}$. ($\mathfrak{f} = \frac{1}{\mathfrak{b}(x-2i)}$)

This effectively introduces old $\mathfrak{A}(x)$ s.t. $\mathfrak{A}(x_j = 0)$ at BAE roots $\{x_j\}$.

$$\frac{\mathfrak{B}(x)}{\mathfrak{F}(x)} = \mathfrak{A}(x+2i) \qquad \frac{\bar{\mathfrak{B}}(x)}{\mathfrak{F}(x)} = \bar{\mathfrak{A}}(x+2i)$$

multiple integral formula III

Theorem (Göhmann et al (2010))

$S = 1$ DME at $T > 0$ has the following multiple integral formula

$$D^{\alpha_1,\dots,\alpha_m}_{\beta_1,\dots,\beta_m}(\xi) = \frac{2^{-m-n_+(\alpha)-n_-(\beta)}}{\prod_{1\le j<k\le m}(\xi_k-\xi_j)^2[(\xi_k-\xi_j)^2+4]}$$

$$\left[\prod_{j=1}^{p}\int_{\mathcal{C}}\frac{d\lambda_j}{2\pi i}F_{z_j}(\lambda_j)\right]\left[\prod_{j=p+1}^{2m}\int_{\overline{\mathcal{C}}}\frac{d\lambda_j}{2\pi i}\overline{F}_{z_j}(\lambda_j)\right]\frac{\det_{2m}\Theta^{(p)}_{j,k}}{\prod_{1\le j<k\le 2m}(\lambda_j-\lambda_k-2i)}$$

Factorization?

- multiple integrals : too complicated to factorize into single loop integrals
- If $\mathfrak{B}(\mu), Y$ already describe physics, only they should appear

Take other routes to find factorized expressions

- use difference equations of $q-$ KZ type at discrete points. (Aufgebauer et al (2012) $S = \frac{1}{2}$)
- fusion of (already factorized) spin $\frac{1}{2}$ DME

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fusion of DME

The idea is trivially simple.

- evaluate D_{2m} of $S = \frac{1}{2}$
- replace $\omega_{\alpha,q}$ of $S = \frac{1}{2}$ to $\omega_{\alpha,q}$ of $S = 1$
- proper combinations of D_{2m} give D_m of $S = 1$ after proper normalization

The actual calculation is simple, but tedious.

$S = 1, m = 3$ result

conevient to present $S = 1$ DME using $SU(2)$ invariant projector

$$D_m^{S=1}(\xi_1, \cdots, \xi_m) = \sum_{\alpha=1}^{N_m} \rho_\alpha^{S=1}(\xi_1, \cdots, \xi_m) P_\alpha^{S=1}$$

$N_1 = 3, N_2 = 15...$

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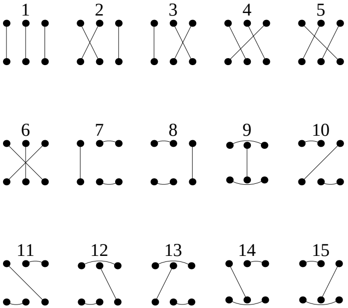
example of projectors for $S = 1$ $m = 3$



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factorized solution ($\xi^{\pm} := \xi \pm i$)

$$\begin{aligned}
\rho_1^{S=1}(\xi_1, \xi_2, \xi_3) &= \frac{1}{27} + \frac{1}{N(\xi_1)N(\xi_2)N(\xi_3)} \Big(c_1^{(1)} \omega(\xi_1^-, \xi_2^-) + c_2^{(1)} \omega(\xi_1^-, \xi_1^+) + c_3^{(1)} \omega(\xi_1^-, \xi_2^+) \\
&+ c_1^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^-) + c_2^{(2)} \omega(\xi_1^-, \xi_2^-)\omega(\xi_2^+, \xi_3^-) + c_3^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^+) \\
&+ c_4^{(2)} \omega(\xi_1^+, \xi_3^-)\omega(\xi_2^-, \xi_3^+) + c_5^{(2)} \omega(\xi_1^-, \xi_2^-)\omega(\xi_1^+, \xi_3^+) + c_6^{(2)} \omega(\xi_2^-, \xi_3^+)\omega(\xi_2^+, \xi_3^-) \\
&+ c_7^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_2^+) + c_8^{(2)} \omega(\xi_2^-, \xi_3^-)\omega(\xi_2^+, \xi_3^+) + c_1^{(3)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^+)\omega(\xi_2^+, \xi_3^-) \\
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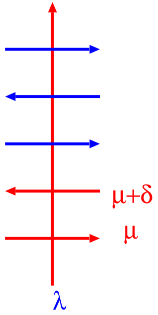
&+ \text{permutations and negation} \Big)
\end{aligned}$$

$S = 1$ result

- $N(\xi)$ comes form normalization
  $N(\xi) = \frac{3}{4} + \frac{\omega(\xi^-, \xi^+)}{2}$.
- $c_j^{(a)}$ are known rational functions of $\xi_k^{\pm}$.
- $\omega(\xi_i^-, \xi_j^+)$ is $(S = \frac{1}{2}) \times (S = 1)$ object

$$\omega(\lambda, \mu) \sim \frac{d}{d\delta} \ln \Lambda^{[1]}(\lambda, \mu)|_{\delta=0}$$

can be obtained using $\mathfrak{b}, y_1$ no need for $\mathfrak{f}$



Almost what we want

homogeneous and $T = 0$ limit of $S = 1$ result

One can take

- zero T limit

$$\omega^{S=1}_{T=0}(\lambda,\mu) = \omega^{S=\frac{1}{2}}_{T=0}(\lambda,\mu) + \frac{(\lambda-\mu)^2+4}{8}\,\frac{\pi(\lambda-\mu)}{2\sinh\frac{\pi}{2}(\lambda-\mu)}$$

- homogeneous limit $\xi_j \to 0$ smoothly

All $\rho^{S=1}_{\alpha}$ are given by rational numbers and $\pi^2, \pi^4 \ldots$,
example

$$8\rho^{S=1}_{1} = \frac{1879}{432} - \frac{3497}{1350}\pi^2 + \frac{53}{135}\pi^4 - \frac{11296}{637875}\pi^6$$

Conjecture (Klümper et al 2013)

The correlation functions at $T = 0$ for the integrable $S = 1$ (integer) spin chain of XXX-type are described by Riemann's ζ functions with even arguments.

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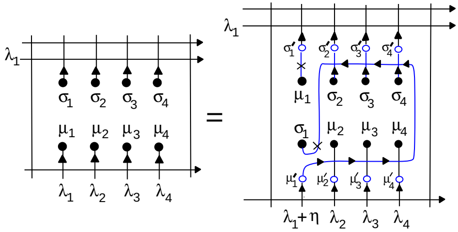
difference equation at discrete points

The difference equation at discrete points (Aufgebauer et al (2012))

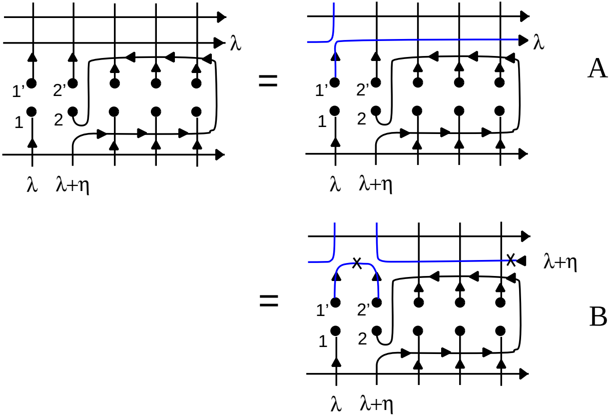


Crosses = charge conjugation operators.
η = crossing parameter $(= 2i)$.
The condition $\lambda_1 \in \{u_1, \cdots, u_N\}$ is absent for q-KZ eq. at $T = 0$ but essential in deriving difference equation for any finite N.

difference equation : derivation 1

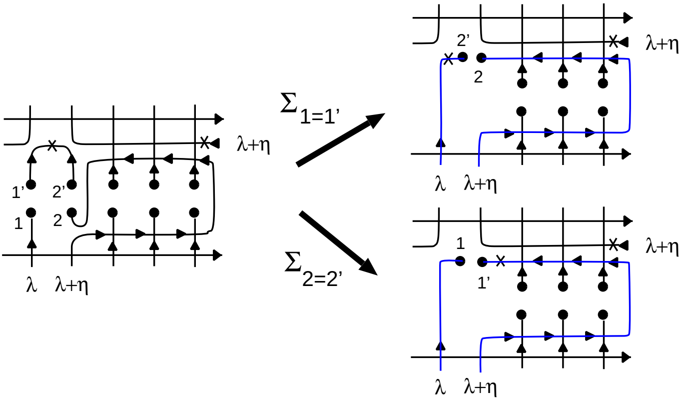
Derivation for $m = 4$.

- step1: start from $m = 5$ with strange configuration.



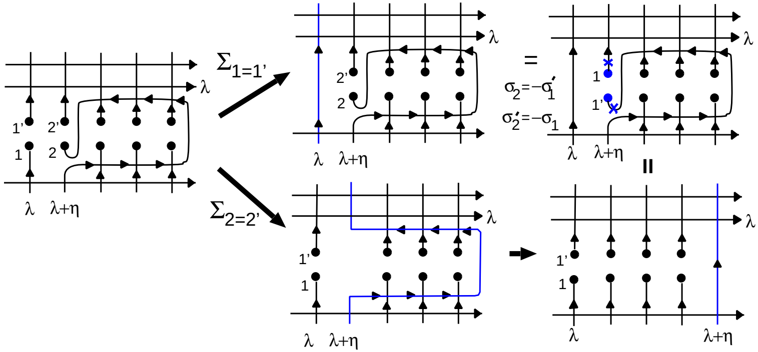
difference equation : derivation 2

- step2: start from Fig. B. Show that $\sum_{\sigma_1=\sigma_1'} \mathrm{FigB} = \sum_{\sigma_2=\sigma_2'} \mathrm{FigB}$ if $\sigma_1 = -\sigma_2', \sigma_2 = -\sigma_1'$.



difference equation : derivation 3

- step2: As Fig.B= Fig. A if $\sigma_1 = -\sigma_2', \sigma_2 = -\sigma_1'$ then $\sum_{\sigma_1=\sigma_1'} \mathrm{FigA} = \sum_{\sigma_2=\sigma_2'} \mathrm{FigA}$. Show this equality is the desired difference equation.



difference equation for ω (Ω)

Thanks to the condition $\xi_j \in \{u_1, \cdots, u_N\}$ one finds closed difference equations for $\omega^S(\lambda, \mu)$.
Define

$$\Omega^{(S)}(\lambda,\mu) = 2i\frac{\omega^{(S)}(\lambda,\mu)+\frac{1}{2}}{(\lambda-\mu)^2+4}$$

$$D_\lambda^{S=\frac{1}{2}}\Omega = -\Omega^{(S)}(\lambda-i,\mu)-\Omega^{(S)}(\lambda+i,\mu)+o(\lambda-\mu)$$

$$o(\lambda) = \frac{2i(\lambda^2-3)}{(\lambda^2+1)(\lambda^2+9)}$$

$$N_1(\lambda) = \frac{1}{1+y_1^{-1}(\lambda)}$$

Then $\Omega^{(S=1)}(\lambda,\mu)$ satisfies the 2nd order difference equation

$$\left(\frac{1}{N_1(\lambda-i)}D_{\lambda-i}^{(\frac{1}{2})}+\frac{1}{N_1(\lambda+i)}D_{\lambda+i}^{(\frac{1}{2})}\right)\Omega(\lambda,\mu) = 0$$

if $\lambda = \xi_j - i$.

difference equation for Ω II

For S general, define recursively

$$D^{\ell}_{\lambda} = \frac{D^{(\ell-\frac{1}{2})}_{\lambda-i}}{N_{2\ell-1}(\lambda-i)} + \frac{D^{(\ell-\frac{1}{2})}_{\lambda+i}}{N_{2\ell-1}(\lambda+i)} - D^{(\ell-1)}_{\lambda} \quad (\ell \geq 1)$$

$$D^{(0)}_{\lambda} = 0$$

$$N_j(\lambda) = \frac{1}{1 + y_j^{-1}(\lambda)}$$

Then $D^{(S)}_{\lambda}\Omega(\lambda,\mu) = 0$ if $\lambda = \xi_j - i$.

difference equation for Ω III

In $T \to 0$ limit, N_j = const.
Let

$$\Omega^{(S)}(\lambda,\mu) = \Omega^{(\frac{1}{2})}(\lambda,\mu) + \Delta^{(S)}(\lambda,\mu)$$

Note

$$D_\lambda^{(\frac{1}{2})}\Omega^{(\frac{1}{2})}(\lambda,\mu) = 0$$

Thus $\Delta^{(S)}(\lambda,\mu)$ satisfies a simpler difference equation (without extra $o(\lambda)$ functions).
consistent with simple results at $T = 0$

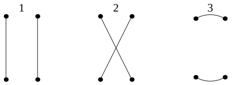
$$\omega_{T=0}^{S=1}(\lambda,\mu) = \omega_{T=0}^{S=\frac{1}{2}}(\lambda,\mu) + \frac{(\lambda-\mu)^2+4}{8}\frac{\pi(\lambda-\mu)}{2\sinh\frac{\pi}{2}(\lambda-\mu)}$$

$$\omega_{T=0}^{S=\frac{3}{2}}(\lambda,\mu) = \omega_{T=0}^{S=\frac{1}{2}}(\lambda,\mu) + \frac{(\lambda-\mu)^2+4}{4}\frac{\pi y_{T=0}}{(1+y_{T=0})\sin\frac{\pi}{5}}\frac{\sinh\frac{\pi}{10}(\lambda-\mu)}{\sinh\frac{\pi}{2}(\lambda-\mu)}$$

$$y_{T=0} = 2\cos\frac{\pi}{5}$$

difference equation for DME at discrete points

Concentrate on $m = 2$: $D_2(\xi_1, \xi_2) = \sum_{\alpha=1}^{3} \rho_\alpha^{S=1}(\xi_1, \xi_2) P_\alpha$



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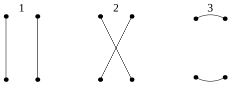

Difference equations

$$\begin{pmatrix} \rho_1(\xi_1 - 2\mathrm{i}, \xi_2) \\ \rho_2(\xi_1 - 2\mathrm{i}, \xi_2) \\ \rho_3(\xi_1 - 2\mathrm{i}, \xi_2) \end{pmatrix} = L(\xi_1 - \xi_2) \cdot \begin{pmatrix} \rho_1(\xi_1, \xi_2) \\ \rho_2(\xi_1, \xi_2) \\ \rho_3(\xi_1, \xi_2) \end{pmatrix}$$

difference equation for DME at discrete points

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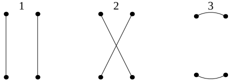


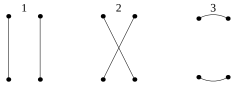
Change of variables

$$\begin{pmatrix} \rho_1(\xi_1,\xi_2) \\ \rho_2(\xi_1,\xi_2) \\ \rho_3(\xi_1,\xi_2) \end{pmatrix} = \begin{pmatrix} \frac{5\xi^2+36}{45(\xi^2+4)} & -\frac{\xi^2}{30(\xi^2+4)} & \frac{\xi^2+6}{15(\xi^2+4)} \\ -\frac{64}{45(\xi^2+4)} & \frac{3\xi^2-20}{60(\xi^2+4)} & -\frac{3\xi^2+28}{30(\xi^2+4)} \\ \frac{16}{45(\xi^2+4)} & \frac{3\xi^2+20}{60(\xi^2+4)} & -\frac{3\xi^2+8}{30(\xi^2+4)} \end{pmatrix} \begin{pmatrix} 1 \\ G(\xi_1,\xi_2) \\ H(\xi_1,\xi_2) \end{pmatrix}$$

$\xi = \xi_1 - \xi_2$

difference equation for DME at discrete points

Concentrate on $m = 2$: $D_2(\xi_1, \xi_2) = \sum_{\alpha=1}^{3} \rho_\alpha^{S=1}(\xi_1, \xi_2) P_\alpha$



Much simpler difference equation

$$\begin{pmatrix} 1 \\ \bar{G} \\ \bar{H} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\xi(\xi-6i)}{(\xi-2i)(\xi+4i)} & 0 \\ -\frac{256i(\xi-i)}{3(\xi+2i)(\xi-2i)^2(\xi+4i)} & -\frac{\xi(\xi-6i)(\xi^2-2i\xi-4)}{(\xi-2i)^2(\xi+2i)(\xi+4i)} & \frac{\xi^2(\xi-6i)(\xi-4i)}{(\xi-2i)^2(\xi+2i)(\xi+4i)} \end{pmatrix} \begin{pmatrix} 1 \\ G \\ H \end{pmatrix}$$

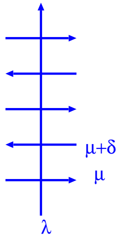
$$\bar{G} = G(\xi_1 - 2i, \xi_2)$$

$$G(\lambda,\mu) \sim (\Omega(\lambda - i, \mu - i) + \Omega(\lambda - i, \mu + i) + \Omega(\lambda + i, \mu - i) + \Omega(\lambda + i, \mu + i))$$

where $\Omega(\lambda,\mu) = 2i\frac{\omega(\lambda,\mu)+1/2)}{(\lambda-\mu)^2+4}$.

- G is expressed by a $(S = 1) \times (S = 1)$ object.
- homogeneous limit is in the physical strip of $\Lambda^{[2]}(\lambda,\mu)$.
- H satisfies difference eq whose source term is G. Thus H is also proper $(S = 1) \times (S = 1)$ object.

$$G(\lambda,\mu) \sim \frac{d}{d\delta} \ln \Lambda^{[2]}(\lambda,\mu)|_{\delta=0}$$



The simplicity of $m = 2$ results at $T = 0$ can be understood from

$$G(\lambda, \mu) \to 0$$

$$H(\lambda, \mu) \to \frac{1}{\sinh^2 \frac{\pi}{2}(\lambda - \mu)}$$

Although they remain non trivial at $T > 0$:
high T expansion results,

$$G(\lambda, \mu) = -\frac{4^2}{3^2} + 256\beta \frac{3\xi^4 + \lambda^2\mu^2\xi^2 + 5(\lambda^4 + \mu^4) + 36\xi^2 + 76(\lambda^2 + \mu^2) + 512}{(\lambda^2 + 4)(\lambda^2 + 16)(\mu^2 + 4)(\mu^2 + 16)} + O(\beta^2)$$

$$\xi = \lambda - \mu$$

$$H(\lambda, \mu) = -\frac{8}{9} + O(\beta)$$

For $S = \frac{3}{2}$,

$$D_2(\xi_1, \xi_2) = \sum_{k=0,3} \rho_k(\xi_1, \xi_2) P_k$$

Fusion gives the f ollowing in $T = 0$ limit.

$$\rho_0 = -\frac{3\left(\sqrt{5}-3\right)\left(40+4\pi^2(\log(4)-1)-35\log(4)\right)}{40\left(7+3\sqrt{5}\right)}$$

$$\rho_1 = \frac{\left(\sqrt{5}-3\right)\left(1830-1455\log(4)+4\pi^2(37\log(4)-47)\right)}{200\left(7+3\sqrt{5}\right)}$$

$$\rho_2 = -\frac{3\left(\sqrt{5}-3\right)\left(715-515\log(4)+\pi^2(52\log(4)-72)\right)}{200\left(7+3\sqrt{5}\right)}$$

$$\rho_3 = \frac{\left(\sqrt{5}-3\right)\left(5635-7770\log(2)+72\pi^2(\log(2048)-8)\right)}{1400\left(7+3\sqrt{5}\right)}$$

Summary and Future problems

Our question was,
can we play the same game for $S > \frac{1}{2}$?

- multiple integral formula at $T = 0$ ✓
- factorization at $T = 0$ ✓
- multiple integral formula at $T > 0$ ✓
- factorization at $T > 0$ ✓
- exponential formula at $T > 0$?
- scaling limit: space of operators in SUSYsG: new operator needed?
- Mixed spin chains?

Thank you for your attention.