Thermal form factors of the XXZ chain
- Zero-temperature limit -

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$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim \sum_n A_n^{-+} \rho_n^m \,,$$

with amplitudes and eigenvalue ratios (correlation lengths)

$$A_n^{-+} = \frac{\langle \Psi_0 | B(0) | \Psi_n \rangle}{\Lambda_n(0) \langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n | C(0) | \Psi_0 \rangle}{\Lambda_0(0) \langle \Psi_n | \Psi_n \rangle} \,, \quad \rho_n = \frac{\Lambda_n(0|0)}{\Lambda_0(0)}$$

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- Analytic expression for the (more general) amplitude:

$$A_n^{-+}(\xi) = \frac{G_+^-(\xi) \overline{G}_-^+(\xi)}{(q^{1+\alpha} - q^{-1-\alpha})(q^\alpha - q^{-\alpha})} \times \exp\left\{ - \int_{\mathcal{C}_n} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} \ln\big(\rho_n(\lambda|\alpha)\big)\, \partial_\lambda \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\} \times \frac{\det_{\mathrm{d}m_+^\alpha, \mathcal{C}_n}\{1 - \widehat{K}_{1-\alpha}\} \det_{\mathrm{d}m_-^\alpha, \mathcal{C}_n}\{1 - \widehat{K}_{1+\alpha}\}}{\det_{\mathrm{d}m_0^\alpha, \mathcal{C}_n}\{1 - \widehat{K}\} \det_{\mathrm{d}m, \mathcal{C}_n}\{1 - \widehat{K}\}}$$

Low-temperature analysis

Consider the form factor series in the critical regime $-1 < \Delta < 1$ at finite magnetic field $h > 0$. In this regime the spectrum of the quantum transfer matrix becomes gapless for $T \to 0$. Consequently,

- infinitely many of the correlation lengths diverge ($\rho_n \to 1$)
- infinitely many terms in the form factor expansion contribute to the leading large-distance asymptotics
- the individual amplitudes must vanish for $T \to 0$

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The low-T analysis of the form factor expansion requires the following steps:

1. Low-temperature analysis of the non-linear integral equations (NLIE)
2. Low-temperature analysis of the eigenvalue ratios
3. Low-temperature analysis of the amplitudes
4. Summation of the leading terms

Low-T analysis of the NLIE

- Set $\eta = -i\gamma$ with $\gamma \in (0, \pi/2]$ such that $\Delta = \mathrm{ch}(\eta) = \cos(\gamma)$ and define

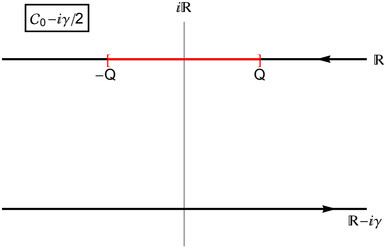
$$\varepsilon_0(\lambda) = h - \frac{4J(1-\Delta^2)}{\mathrm{ch}(2\lambda) - \Delta}, \quad K(\lambda) = \mathrm{cth}(\lambda - \eta) - \mathrm{cth}(\lambda + \eta)$$

$$\theta(\lambda) = \ln\left(\frac{\mathrm{sh}(\eta - \lambda)}{\mathrm{sh}(\eta + \lambda)}\right), \quad u(\lambda) = -T \ln\big(\mathfrak{a}_n(\lambda + \mathrm{i}\gamma/2|\alpha)\big)$$

Then the NLIE for the auxiliary function turns into

$$u(\lambda) = \varepsilon_0(\lambda) + T\left[2\pi\mathrm{i}\left(\alpha' - \tfrac{1}{2}\right) + \sum_{j=1}^{n'} \theta(\lambda - \lambda_j^p + \mathrm{i}\gamma/2) - \sum_{j=1}^{n'+1} \theta(\lambda - \lambda_j^h + \mathrm{i}\gamma/2)\right] + T\int_{\mathcal{C}_0 - \mathrm{i}\gamma/2} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\, K(\lambda - \mu) \ln\left(1 + \mathrm{e}^{-\frac{u(\mu)}{T}}\right)$$

where $\alpha' = \eta\alpha/\mathrm{i}\pi$, where the extra contributions come from straightening the contour and where we assumed that $n' = n_p = n_h - 1$.



- Define $\varepsilon := \lim_{T \to 0} u$. For $T \to 0$ we have

$$-T \ln\left(1 + \mathrm{e}^{-\frac{u(\lambda)}{T}}\right) \to \begin{cases} 0 & \text{if } \mathrm{Re}\,\varepsilon(\lambda) > 0 \\ \varepsilon(\lambda) & \text{if } \mathrm{Re}\,\varepsilon(\lambda) < 0 \end{cases}$$

For $T \to 0$ the integrand vanishes on those parts of the contour on which $\mathrm{Re}\,\varepsilon > 0$ and is nonzero on their complement. We claim that this complement is an interval $[-Q, Q]$ on the real axis. Indeed, if

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) + \int_{-Q}^{Q} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \, K(\lambda - \mu)\varepsilon(\mu) \, ,$$

where Q is determined by $\varepsilon(\pm Q) = 0$, then $\varepsilon(\lambda) < 0$ for $\lambda \in [-Q, Q]$ and $\mathrm{Re}\,\varepsilon(\lambda) > 0$ for $\lambda \in \mathcal{C}_0 - \mathrm{i}\gamma/2 \setminus [-Q, Q]$.

Sommerfeld lemma

Argument becomes rigorous by employing the 'Sommerfeld lemma':

- Let u, f be holomorphic in an open set containing a contour $\mathcal{C}_u$, and let f be bounded on $\mathcal{C}_u$. Let $v = \mathrm{Re}\, u$, $w = \mathrm{Im}\, u$. Assume that v has exactly two zeros $Q_\pm$ on $\mathcal{C}_u$ separating $\mathcal{C}_u$ into a part $\mathcal{C}_u^-$ between Q_- and Q_+ on which v is negative and a remainder $\mathcal{C}_u^+$ on which v is positive. Assume that $\exists p \in \mathbb{Z}$ such that $w(Q_\pm) = 2\pi p T$. Assume that $\mathcal{C}_u$ is oriented in such a way that Q_- comes before Q_+ on $\mathcal{C}_u^-$. Then (for $T > 0$)

$$T \int_{\mathcal{C}_u} \mathrm{d}\lambda\, f(\lambda) \ln\Big(1 + \mathrm{e}^{-\frac{u(\lambda)}{T}}\Big) = -\int_{Q_-}^{Q_+} \mathrm{d}\lambda\, f(\lambda)\big(u(\lambda) - 2\pi \mathrm{i} p T\big) + \frac{T^2\pi^2}{6}\left(\frac{f(Q_+)}{u'(Q_+)} - \frac{f(Q_-)}{u'(Q_-)}\right) + \mathcal{O}(T^4)\,.$$

- allows for the calculation of first- and second-order T-corrections of u

Locus of Bethe roots at low temperature

- Asymptotic form of Bethe equations

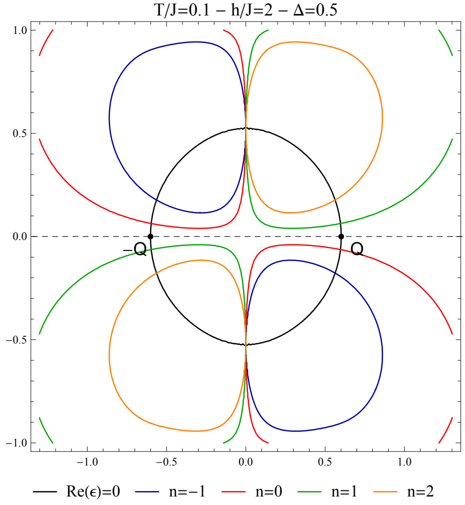
$$\ln \mathfrak{a}(\lambda) \sim -\frac{\varepsilon(\lambda - \mathrm{i}\gamma/2)}{T} = (2n-1)\mathrm{i}\pi$$

for $n \in \mathbb{Z}$

- Figure shows curves

$$\mathrm{Re}\,\varepsilon(\lambda) = 0$$
$$\mathrm{Im}\,\varepsilon(\lambda) = (2n-1)\pi T$$

Intersections are Bethe roots or holes
$\Rightarrow$ roots and holes accumulate at $\pm Q$ for $T \to 0$



- First order correction for u is given by

$$u(\lambda) = \varepsilon(\lambda) + u_1^{(\ell)}(\lambda)T + \mathcal{O}(T^2)\ , \quad u_1^{(\ell)}(\lambda) = 2\pi\mathrm{i}\Big(\big(\alpha' - \ell - \tfrac{1}{2}\big)Z(\lambda) + \phi(\lambda, Q) + \ell\Big)$$

Here $\ell = n_h^- - n_p^- = n_p^+ - n_h^+ + 1$ (number of Umklapp processes) and

$$Z(\lambda) = 1 + \int_{-Q}^{Q} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} K(\lambda - \mu) Z(\mu)\,,$$

$$\phi(\lambda, \nu) = -\frac{\theta(\lambda - \nu)}{2\pi\mathrm{i}} + \int_{-Q}^{Q} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} K(\lambda - \mu)\phi(\mu, \nu)\,,$$

$$\rho(\lambda) = -\frac{\mathrm{e}(\lambda + \mathrm{i}\gamma/2)}{2\pi\mathrm{i}} + \int_{-Q}^{Q} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} K(\lambda - \mu)\rho(\mu)$$

Z dressed charge function, ϕ dressed phase, ρ root density

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- Fundamental characteristic parameters

$$\mathcal{Z} = Z(Q) \qquad \text{'dressed charge'}$$

$$k_F = \pi \int_{-Q}^{Q} \mathrm{d}\lambda\, \rho(\lambda) \qquad \text{'Fermi momentum'}$$

$$v_0 = \frac{\varepsilon'(Q)}{2\pi\rho(Q)} \qquad \text{'sound velocity'}$$

Low-T analysis of correlation lengths

Using the Sommerfeld lemma we obtain

- The eigenvalue ratios (resp. correlation lengths)

$$\rho_n(0|\alpha) = q^\alpha \exp\Big\{ \mathrm{i}\pi - 2\mathrm{i}(\alpha' - \ell)k_F - \frac{2\pi T}{v_0}\Big[(\alpha' - \ell)^2 \mathcal{Z}^2 + \frac{1}{4\mathcal{Z}^2} - \ell^2 + \ell - 1 + \sum_{j=1}^{n'+1} h_j + \sum_{j=1}^{n'} (p_j - 1) \Big] \Big\} + \mathcal{O}(T^2)$$

where $\{h_j\} = \{h_j^+\}_{j=1}^{n_h^+} \cup \{h_j^-\}_{j=1}^{n_h^-} \subset \mathbb{N}^{n'+1}$, $\{p_j\} = \{p_j^+\}_{j=1}^{n_p^+} \cup \{p_j^-\}_{j=1}^{n_p^-} \subset \mathbb{N}^{n'}$, and $h_j \neq h_k$ for $j \neq k$, $p_j \neq p_k$ for $j \neq k$.

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- h_j and p_k are quantum numbers that parameterize the particle and hole rapidities at the left and right Fermi edge

$$\lambda_j^{h^\pm} - \mathrm{i}\gamma/2 = x_{h_j}^{\pm} = \pm Q - \frac{2\pi\mathrm{i} T}{\varepsilon'(Q)} \Bigg\{ h_j^{\pm} - 1/2 \pm \frac{u_1^{(\ell)}(\pm Q)}{2\pi\mathrm{i}} \Bigg\} + \mathcal{O}(T^2)$$

$$\lambda_j^{p^\pm} - \mathrm{i}\gamma/2 = y_{p_j}^{\pm} = \pm Q + \frac{2\pi\mathrm{i} T}{\varepsilon'(Q)} \Bigg\{ p_j^{\pm} - 1/2 \mp \frac{u_1^{(\ell)}(\pm Q)}{2\pi\mathrm{i}} \Bigg\} + \mathcal{O}(T^2)$$

Low-T analysis of the amplitudes - Determinants

- The amplitudes consist of a determinant factor times $G_+^-(\xi)\overline{G}_-^+(\xi)$ times the 'universal factor'

$$A_0^{(n)}(\alpha) = \exp\left\{ -\int_{\mathcal{C}_n} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \ln\bigl(\rho_n(\lambda|\alpha)\bigr)\, \partial_\lambda \ln\left(\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{1+\mathfrak{a}_0(\lambda)} \right) \right\}$$

- Determinants in the denominator:

$$\lim_{T\to 0+} \det_{\mathrm{d}m_0^\alpha, \mathcal{C}_n} \bigl\{1 - \widehat{K}\bigr\} = \lim_{T\to 0+} \det_{\mathrm{d}m, \mathcal{C}_n} \bigl\{1 - \widehat{K}\bigr\} = \det_{\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}, [-Q,Q]} \bigl\{1 - \widehat{K}\bigr\}$$

since $\bigl(1+\mathfrak{a}_0^{-1}(\lambda)\bigr)^{-1}$ and $\bigl(1+\mathfrak{a}_n^{-1}(\lambda|\alpha)\bigr)^{-1}$ turn into the characteristic functions of the 'interval' $\mathrm{i}\gamma/2 + [-Q, Q]$ for $T \to 0$.

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$$\frac{1}{\rho_n(\lambda|\alpha)(1+\mathfrak{a}_0(\lambda))} = \frac{-q^{-\alpha}\phi(\lambda+\eta)}{\rho_n(\lambda|\alpha)\big(q^{\alpha}\phi(\lambda-\eta)-q^{-\alpha}\phi(\lambda+\eta)\big)} + \frac{\phi(\lambda)}{q^{\alpha}\phi(\lambda-\eta)-q^{-\alpha}\phi(\lambda+\eta)}$$

with $\phi(\lambda) = \prod_{j=1}^{M-1} \mathrm{sh}(\lambda-\mu_j) \Big/ \prod_{j=1}^{M} \mathrm{sh}(\lambda-\lambda_j)$

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- We argue that the first term can be dropped:
  The function

$$\frac{-q^{-\alpha}\phi(\lambda+\eta)}{q^{\alpha}\phi(\lambda-\eta)-q^{-\alpha}\phi(\lambda+\eta)} = \left[1-\frac{\mathfrak{a}_0(\lambda)}{\mathfrak{a}_n(\lambda)}\right]^{-1} = \Big[1-\exp\big\{u_1^{(\ell)}(\lambda-\mathrm{i}\gamma/2)\big\}+\mathcal{O}(T)\Big]^{-1}$$

has no poles inside the strip $|\,\mathrm{Im}(\lambda)| \leq \gamma/2$ (can be verified at least numerically)

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- Perform limit $T\to 0$ for λ away from $\pm Q$

$$\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\frac{\phi(\lambda)}{q^{\alpha}\phi(\lambda-\eta)-q^{-\alpha}\phi(\lambda+\eta)} = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{\rho_n(\lambda|\alpha)(1+\mathfrak{a}_0(\lambda))}\,\frac{1}{1-\mathfrak{a}_n(\lambda|\alpha)/\mathfrak{a}_0(\lambda)}$$

$$\xrightarrow{T\to 0}\ \mathrm{d}\widehat{M}_{-}^{\alpha}(\lambda-\mathrm{i}\gamma/2)$$

- The new measures $\mathrm{d}\widehat{M}^{\alpha}_{\pm}$ read

$$\mathrm{d}\widehat{M}^{\alpha}_{\pm}(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \frac{\exp\Big\{ \pm \mathrm{i}\pi\alpha' \pm \mathsf{E}(Q-\lambda) \pm \int_{-Q}^{Q} \mathrm{d}\mu \, \mathrm{e}(\mu-\lambda)\big(\frac{u_1^{(\ell)}(\mu)}{2\pi\mathrm{i}} - \ell\big)\Big\}}{1 - \exp\big\{ \pm u_1^{(\ell)}(\lambda)\big\}}$$

- Last step: Shrink contour $\Gamma_n^{(-)} \to \Gamma[-Q, Q] + \mathrm{i}\gamma/2$ and shift down to the real axis

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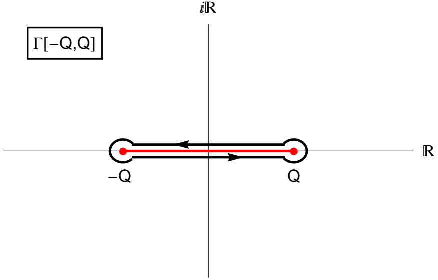

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- It follows that the zero-temperature limit of the determinant part is

$$\mathcal{D}(\ell) = \frac{\det_{\mathrm{d}\widehat{M}^{\alpha}_{+},\Gamma[-Q,Q]}\big\{1 - \widehat{K}_{1-\alpha}\big\} \det_{\mathrm{d}\widehat{M}^{\alpha}_{-},\Gamma[-Q,Q]}\big\{1 - \widehat{K}_{1+\alpha}\big\}}{\det^2_{\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}},[-Q,Q]}\big\{1 - \widehat{K}\big\}} .$$

- $\mathcal{D}$ depends only on ℓ, not on the 'quantum numbers' $h_j^{\pm}$, $p_j^{\pm}$

- Similarly, one can treat the 'factorizing part'

$$\mathcal{G}(\ell) = \lim_{\substack{\alpha\to 0\\ \xi\to 0}} \lim_{T\to 0} \frac{G_{+}^{-}(\xi)\overline{G}_{-}^{+}(\xi)}{(q^{1+\alpha} - q^{-1-\alpha})(q^{\alpha} - q^{-\alpha})}$$

Low-T analysis of the amplitudes - Universal part

- Last step: the 'universal part'

$$A_0^{(n)}(\alpha) = \exp\left\{ -\int_{\mathcal{C}_n} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} \ln\bigl(\rho_n(\lambda|\alpha)\bigr)\, \partial_\lambda \ln\left(\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{1+\mathfrak{a}_0(\lambda)}\right)\right\}$$

- Low-T analysis of $A_0^{(n)}$ cumbersome because of singular integrals
- Calculation possible by methods similar to those developed by Kozlowski, Maillet, Slavnov (2011) for the Bose gas

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- Low-T analysis of $A_0^{(n)}$ cumbersome because of singular integrals
- Calculation possible by methods similar to those developed by Kozlowski, Maillet, Slavnov (2011) for the Bose gas
- Result:

$$A_0^{(n)}(\alpha) = \frac{(-1)^{\ell}\pi\,\mathrm{sh}(2Q)}{\sin(\gamma)\sin\left(\pi\frac{u_1^{(\ell)}(Q)}{2\pi\mathrm{i}}\right)} A_n^{(-)}(\alpha) A_n^{(+)}(\alpha)$$

where (for $\epsilon = \pm 1$)

$$A_n^{(\epsilon)}(\alpha) = \exp\left(\tfrac{1}{2}\,\mathrm{C}\left[\frac{u_1^{(\ell)}}{2\pi\mathrm{i}} - \ell\right]\right) \sin\left(\pi\frac{u_1^{(\ell)}(\epsilon Q)}{2\pi\mathrm{i}}\right)^{\epsilon\ell} \left(\frac{2\pi T}{\epsilon'(Q)\,\mathrm{sh}(2Q)}\right)^{(\alpha'-\ell)^2\mathcal{Z}^2 + \frac{1}{4\mathcal{Z}^2}}$$
$$\times\, \mathrm{G}\left(1+\frac{u_1^{(\ell)}(\epsilon Q)}{2\pi\mathrm{i}}\right) \mathrm{G}\left(1-\frac{u_1^{(\ell)}(\epsilon Q)}{2\pi\mathrm{i}}\right) \left(\frac{1}{\pi}\sin\left(\pi\frac{u_1^{(\ell)}(\epsilon Q)}{2\pi\mathrm{i}}\right)\right)^{2n_h^{\epsilon}} \mathcal{R}_{n_h^{\varepsilon}, n_p^{\varepsilon}}\left(\{h_j^{\varepsilon}\}, \{p_j^{\varepsilon}\}\,\middle|\, \varepsilon\frac{u_1^{(\ell)}(\epsilon Q)}{2\pi\mathrm{i}}\right)$$

- Same scaling behaviour as for the critical finite-size form factors Kitanine et al. (2009)

- We have introduced the shorthand notation

$$C[v]=\int_{-Q}^{Q}\mathrm{d}\lambda\int_{-Q}^{Q}\mathrm{d}\mu\left[\frac{v'(\lambda)v(\mu)-v'(\mu)v(\lambda)}{2\,\mathrm{th}(\lambda-\mu)}-\frac{v(\lambda)v(\mu)}{\mathrm{sh}^2(\lambda-\mu+\eta)}\right]$$
$$+\,(v(Q)+2)\int_{-Q}^{Q}\mathrm{d}\lambda\,\frac{v(\lambda)-v(Q)}{\mathrm{th}(\lambda-Q)}-\int_{-Q}^{Q}\mathrm{d}\lambda\left[\frac{v(\lambda)}{\mathrm{th}(\lambda-Q+\eta)}+\frac{v(\lambda)}{\mathrm{th}(\lambda-Q-\eta)}\right]$$

and

$$\mathcal{R}_{n_1,n_2}\big(\{h_j\},\{p_j\}\big|v\big)=\frac{\prod_{1\le j<k\le n_1}(h_j-h_k)^2\prod_{1\le j<k\le n_2}(p_j-p_k)^2}{\prod_{j=1}^{n_1}\prod_{k=1}^{n_2}(h_j+p_k-1)^2}$$
$$\times\left[\prod_{j=1}^{n_1}\frac{\Gamma^2(h_j+v)}{\Gamma^2(h_j)}\right]\left[\prod_{j=1}^{n_2}\frac{\Gamma^2(p_j-v)}{\Gamma^2(p_j)}\right]$$

- We have introduced the shorthand notation

$$C[v] = \int_{-Q}^{Q} \mathrm{d}\lambda \int_{-Q}^{Q} \mathrm{d}\mu \left[\frac{v'(\lambda)v(\mu) - v'(\mu)v(\lambda)}{2\,\mathrm{th}(\lambda-\mu)} - \frac{v(\lambda)v(\mu)}{\mathrm{sh}^2(\lambda-\mu+\eta)} \right] + (v(Q)+2)\int_{-Q}^{Q} \mathrm{d}\lambda\, \frac{v(\lambda)-v(Q)}{\mathrm{th}(\lambda-Q)} - \int_{-Q}^{Q} \mathrm{d}\lambda \left[\frac{v(\lambda)}{\mathrm{th}(\lambda-Q+\eta)} + \frac{v(\lambda)}{\mathrm{th}(\lambda-Q-\eta)} \right]$$

and

$$\mathcal{R}_{n_1,n_2}\big(\{h_j\},\{p_j\}\big|v\big) = \frac{\prod_{1\le j<k\le n_1}(h_j-h_k)^2 \prod_{1\le j<k\le n_2}(p_j-p_k)^2}{\prod_{j=1}^{n_1}\prod_{k=1}^{n_2}(h_j+p_k-1)^2} \times \left[\prod_{j=1}^{n_1} \frac{\Gamma^2(h_j+v)}{\Gamma^2(h_j)} \right] \left[\prod_{j=1}^{n_2} \frac{\Gamma^2(p_j-v)}{\Gamma^2(p_j)} \right]$$

- Recall that $u_1^{(\ell)}(\lambda) = 2\pi\mathrm{i}\Big(\big(\alpha' - \ell - \frac{1}{2}\big) Z(\lambda) + \phi(\lambda, Q) + \ell \Big)$

  Thus, in the low-temperature limit we have have obtained an entirely explicit description of the 'universal contributions' $A_n^{(0)}$ to the amplitudes in terms of the dressed charge, the dressed energy and the dressed phase. Recall that the above formulae are valid for a certain class of excitations characterized by (i) $n_p = n_h - 1$, and (ii) $x_{h_j}^{\pm}, y_{p_j}^{\pm} = \pm Q + \mathcal{O}(T)$.

Summation

- Critical form factor summation formula

$$\sum_{\substack{n_p,n_h\geq 0\\ n_p-n_h=\ell}} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_j\in\mathbb{N}}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_j\in\mathbb{N}}} \exp\left[\tfrac{-2\pi m T}{v_0}\left(\textstyle\sum_{j=1}^{n_p}(p_j-1)+\sum_{j=1}^{n_h} h_j\right)\right]$$

$$\times\left(\frac{\sin(\pi v)}{\pi}\right)^{2n_h}\mathcal{R}_{n_h,n_p}\left(\{h_j\},\{p_j\}\middle|v\right)=\frac{G^2(1+\ell-v)}{G^2(1-v)}\,\frac{\exp\left[\frac{-\pi m T\ell(\ell-1)}{v_0}\right]}{\left(1-\exp\left[\frac{-2\pi m T}{v_0}\right]\right)^{(\ell-v)^2}}$$

(Kerov et al. 1993, Kitanine et al. 2011)

Large-distance asymptotics for low temperatures

- Summing up all the contributions of the form considered above we obtain

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim \sum_{\ell=-\infty}^{\infty} (-1)^m \, \mathrm{e}^{2\mathrm{i} k_F \ell m} \, \mathcal{D}(\ell) \, \mathcal{G}(\ell) \, \mathcal{A}(\ell) \left(\frac{\pi T / v_0}{\mathrm{sh}(m \pi T / v_0)} \right)^{2\ell^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}}$$

where

$$\mathcal{A}(\ell) = \frac{\mathrm{sh}(2Q)}{-\sin(\gamma)} \frac{\exp\left(C \left[\frac{u_1^{(\ell)}}{2\pi \mathrm{i}} - \ell \right] \right) \prod_{\epsilon_1 = \pm 1, \epsilon_2 = \pm 1} G\left(1 + \epsilon_1 \ell \mathcal{Z} + \epsilon_2 \frac{1}{2\mathcal{Z}}\right)}{\left(2\pi \rho(Q) \, \mathrm{sh}(2Q)\right)^{2\ell^2 \mathcal{Z}^2 + 1/(2\mathcal{Z}^2)}}$$

- The leading asymptotics ($\ell = 0$) is in accordance with Conformal Field Theory (or Tomonaga-Luttinger liquid theory) Luther, Peschel (1975), Haldane (1981)
- For $T \to 0$ the individual terms in the sum show algebraic decay
- For $T \to 0$ the sum coincides with the result obtained by Kitanine et al. (2011)

Limit of zero magnetic field

- Exact result for the $\ell = 0$ amplitude A_0^{-+} by Lukyanov (1999) for zero magnetic field:

$$A_0^{-+}(h=0) = \frac{\pi^2}{4\gamma^2}\left[\frac{\Gamma(\frac{\pi-\gamma}{2\gamma})}{2\sqrt{\pi}\,\Gamma(\frac{\pi}{2\gamma})}\right]^{\frac{\pi-\gamma}{\pi}} \exp\left\{-\int_0^\infty \frac{\mathrm{d}t}{t}\left(\frac{\mathrm{sh}(\frac{\pi-\gamma}{\pi}t)}{\mathrm{sh}(t)\,\mathrm{ch}(\frac{\gamma}{\pi}t)} - \frac{\pi-\gamma}{\pi}\,\mathrm{e}^{-2t}\right)\right\}$$

- Open problem: How to obtain this *analytically* from our results?
- Numerics indicate that the limit $h \to 0$ exists and is indeed given by Lukyanov's formula

Numerics

- Leading asymptotics given by $\ell = 0$

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim (-1)^m \underbrace{\mathcal{D}(0)\,\mathcal{G}(0)\,\mathcal{A}(0)}_{A_0^{-+}} \left(\frac{\pi T / v_0}{\mathrm{sh}(m\pi T / v_0)} \right)^{\frac{1}{2\mathcal{Z}^2}}$$

- All quantities defined by linear integral equations (e.g. $Z, \epsilon, \phi, \ldots$) easily and accurately computable

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- All quantities defined by linear integral equations (e.g. $Z, \epsilon, \phi, \ldots$) easily and accurately computable
- Determinants in the numerator?
  Calculate

$$\mathrm{d}\widehat{M}^0_{\pm}(\lambda_-) - \mathrm{d}\widehat{M}^0_{\pm}(\lambda_+) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \left[\frac{\mathrm{sh}(Q-\lambda)\,\mathrm{sh}(Q+\lambda-\eta)}{\mathrm{sh}(Q+\lambda)\,\mathrm{sh}(\lambda-Q-\eta)} \right]^{\pm u_1^{(0)}(\lambda)/2\pi\mathrm{i}}$$
$$\times \exp\left[\pm \mathrm{E}(Q-\lambda) \mp \mathrm{i}\pi \frac{u_1^{(0)}(\lambda)}{2\pi\mathrm{i}} \pm \int_{-Q}^{Q} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\ \mathrm{e}(\mu-\lambda)\left(u_1^{(0)}(\mu) - u_1^{(0)}(\lambda)\right) \right]$$

- Weight function behaves like $(Q+\lambda)^{\pm 1/(2\mathcal{Z})}$ for $\lambda \sim -Q$ and $(Q-\lambda)^{\pm 1/(2\mathcal{Z})}$ for $\lambda \sim Q$
  $\Rightarrow$ Integrable singularities at $\pm Q$ $\Rightarrow$ Integral operator acting on $[-Q, Q]$

Numerical results

$A_0^{-+} = A_0^{-+}(\Delta, M)$ with $M = \frac{1}{2}\langle\sigma_1^z\rangle = \frac{1}{2} - \frac{k_F}{\pi}$ being the magnetization

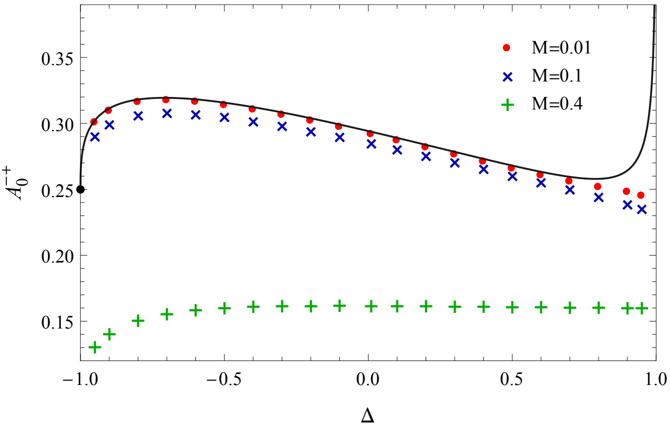

Figure: Amplitude A_0^{-+} vs. anisotropy Δ for different values of the magnetization M. Exact result for $M = 0$ by LUKYANOV is marked by the solid line.

Numerical results

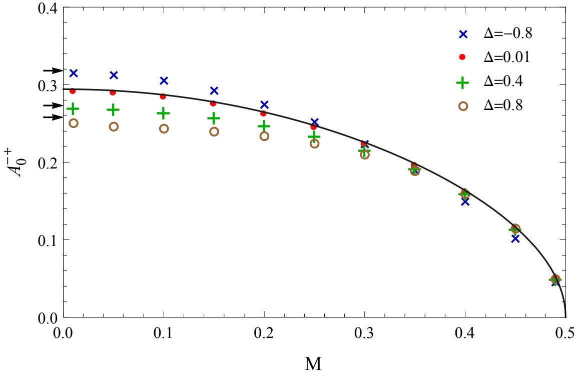

Figure: Amplitude A_0^{-+} vs. magnetization M for different values of Δ. Exact result by OVCHINNIKOV is marked by the solid line. Exact values for $M = 0$ are indicated by arrows.

- Results in agreement with analytical [LUKYANOV (1999), OVCHINNIKOV (2007)] and numerical results [HIKIHARA, FURUSAKI (2004) DMRG, SHASHI ET AL. (2012) field theory + finite size scaling]

Summary and Outlook

- We have calculated explicitly the leading low-temperature asymptotics of the thermal form factors
- Summation of the form factor series confirms predictions of CFT and TLL theory concerning the large-distance asymptotics of the transversal correlation functions
- In addition, summation provides the non-universal amplitudes as functions of the magnetic field. These can be efficiently evaluated and numerically agree with known results for $h \to 0$.
- The low-T analysis is a non-trivial test for the validity of our general formulae

Open questions:

- Amplitudes for zero magnetic field?
- Form factors in the massive regime $\Delta > 1$ at $T = 0$
  (cf. JIMBO, MIWA (1995) for $h = 0$)