Multiple integral formulae for $SU(3)$ on/off-shell scalar product

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Outline

1. Definition of the scalar product in $SU(3)$-invariant models.
2. Multiple integral formulae for $SU(3)$ on/off-shell scalar product.
3. Proof of (the first of the) multiple integral formulae.
4. Recovering known results as limiting cases.

Some notation

- We use boldface, with a subscript, to denote a set and its cardinality:

$$\boldsymbol{x}_m = \{x_1, \ldots, x_m\}, \qquad \boldsymbol{y}_n = \{y_1, \ldots, y_n\}$$

  Sometimes the subscript can be omitted when the cardinality is clear from context, *e.g.* $\boldsymbol{x}_m \equiv \boldsymbol{x}$.

- Omission of an element is indicated by a circumflex and an additional subscript:

$$\widehat{\boldsymbol{x}}_{m,i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}, \qquad \widehat{\boldsymbol{y}}_{n,j} = \{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n\}$$

  At times we keep only the subscript of the omitted variable, *e.g.* $\widehat{\boldsymbol{x}}_{m,i} \equiv \widehat{\boldsymbol{x}}_i$.

- We use $\oplus$ to denote a union of sets:

$$\boldsymbol{x}_m \oplus \boldsymbol{y}_n = \{x_1, \ldots, x_m\} \oplus \{y_1, \ldots, y_n\} = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$

- We use $\ominus$ to denote exclusion of a subset:

$$\{x_1, \ldots, x_m, y_1, \ldots, y_n\} \ominus \boldsymbol{y}_n = \{x_1, \ldots, x_m\}$$

Useful functions

- We define three types of rational function:

$$f(x,y) = \frac{x-y+1}{x-y}, \qquad g(x,y) = \frac{1}{x-y}, \qquad h(x,y) = x-y+1$$

- When these functions take a set as an argument, a product over all elements in the set is implied:

$$f(x,\boldsymbol{y}_n) = \prod_{j=1}^{n} f(x,y_j), \qquad f(\boldsymbol{x}_m,y) = \prod_{i=1}^{m} f(x_i,y), \qquad f(\boldsymbol{x}_m,\boldsymbol{y}_n) = \prod_{i=1}^{m}\prod_{j=1}^{n} f(x_i,y_j)$$

- Combining all of this notation, we have (for example)

$$f(w,\boldsymbol{x}_\ell \oplus \boldsymbol{y}_m \ominus \boldsymbol{z}_n) = \frac{\prod_{i=1}^{\ell} f(w,x_i)\prod_{j=1}^{m} f(w,y_j)}{\prod_{k=1}^{n} f(w,z_k)}$$

which is well defined, even if $\boldsymbol{z}_n$ is not a subset of $\boldsymbol{x}_\ell \oplus \boldsymbol{y}_m$.

$SU(3)$-invariant models

- The $SU(3)$-invariant R-matrix is given by

$$R^{(1)}_{\alpha\beta}(\lambda,\mu) =$$

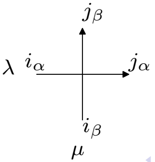
$$\left(\begin{array}{ccc|ccc|ccc}
f(\lambda,\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & g(\lambda,\mu) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g(\lambda,\mu) & 0 & 0 \\
\hline
0 & g(\lambda,\mu) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f(\lambda,\mu) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & g(\lambda,\mu) & 0 \\
\hline
0 & 0 & g(\lambda,\mu) & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g(\lambda,\mu) & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\lambda,\mu)
\end{array}\right)_{\alpha\beta}$$

- The $SU(2)$-invariant R-matrix is given by

$$R^{(2)}_{\alpha\beta}(\lambda,\mu) = \left(\begin{array}{cc|cc}
f(\lambda,\mu) & 0 & 0 & 0 \\
0 & 1 & g(\lambda,\mu) & 0 \\
\hline
0 & g(\lambda,\mu) & 1 & 0 \\
0 & 0 & 0 & f(\lambda,\mu)
\end{array}\right)_{\alpha\beta}$$

- The entries of either R-matrix have the graphical representation

$$\left[R^{(n)}_{\alpha\beta}(\lambda,\mu)\right]^{i_\alpha j_\alpha}_{i_\beta j_\beta} =$$



$SU(3)$-invariant models

- Consider a family of operators, grouped in the monodromy matrix

$$T^{(1)}_{\alpha}(\lambda) = \left(\begin{array}{ccc} T_{11}(\lambda) & T_{12}(\lambda) & T_{13}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) & T_{23}(\lambda) \\ T_{31}(\lambda) & T_{32}(\lambda) & T_{33}(\lambda) \end{array} \right)_{\alpha}$$

whose commutation relations are prescribed by the bilinear relation

$$R^{(1)}_{\alpha\beta}(\lambda,\mu) T^{(1)}_{\alpha}(\lambda) T^{(1)}_{\beta}(\mu) = T^{(1)}_{\beta}(\mu) T^{(1)}_{\alpha}(\lambda) R^{(1)}_{\alpha\beta}(\lambda,\mu)$$

- Construct Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^*$ by assuming the following action of the operators on pseudo-vacuum states $|0\rangle$ and $\langle 0|$:

$$\left. \begin{array}{lll} T_{ii}(\lambda)|0\rangle = a_i(\lambda)|0\rangle, & T_{kj}(\lambda)|0\rangle = 0, & T_{jk}(\lambda)|0\rangle \neq 0 \\ \langle 0|T_{ii}(\lambda) = a_i(\lambda)\langle 0|, & \langle 0|T_{kj}(\lambda) \neq 0, & \langle 0|T_{jk}(\lambda) = 0 \end{array} \right\} \forall \begin{array}{c} 1 \leqslant i \leqslant 3 \\ 1 \leqslant j < k \leqslant 3 \end{array}$$

- The Bethe Ansatz allows us to find the eigenvectors and eigenvalues of the transfer matrix:

$$\mathcal{T}(\lambda) = \sum_{k=1}^{3} T_{kk}(\lambda)$$

Nested Bethe Ansatz [Kulish, Reshetikhin 83] [Belliard, Ragoucy 08]

- Take the monodromy matrix

$$T^{(1)}_{\alpha}(\lambda) = \left(\begin{array}{ccc} T_{11}(\lambda) & T_{12}(\lambda) & T_{13}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) & T_{23}(\lambda) \\ T_{31}(\lambda) & T_{32}(\lambda) & T_{33}(\lambda) \end{array} \right)_{\alpha}$$

and break it into sub-matrices:

$$B^{(1)}_{\beta}(\lambda) = \left[\begin{array}{c} T_{21}(\lambda) \\ T_{31}(\lambda) \end{array} \right]_{\beta} \quad C^{(1)}_{\gamma}(\lambda) = \left[\begin{array}{cc} T_{12}(\lambda) & T_{13}(\lambda) \end{array} \right]_{\gamma} \quad D^{(1)}_{\delta}(\lambda) = \left[\begin{array}{cc} T_{22}(\lambda) & T_{23}(\lambda) \\ T_{32}(\lambda) & T_{33}(\lambda) \end{array} \right]_{\delta}$$

- Repeat this for the $SU(2)$-type monodromy matrices below:

$$\begin{aligned} T^{(2)}_{\delta}(\mu|\lambda_{\ell},\dots,\lambda_1) &= D^{(1)}_{\delta}(\mu) R^{(2)}_{\delta\alpha_{\ell}}(\mu,\lambda_{\ell}) \dots R^{(2)}_{\delta\alpha_1}(\mu,\lambda_1) \\ &= \left(\begin{array}{cc} A^{(2)}(\mu|\lambda_{\ell},\dots,\lambda_1) & C^{(2)}(\mu|\lambda_{\ell},\dots,\lambda_1) \\ B^{(2)}(\mu|\lambda_{\ell},\dots,\lambda_1) & D^{(2)}(\mu|\lambda_{\ell},\dots,\lambda_1) \end{array} \right)_{\delta} \end{aligned}$$

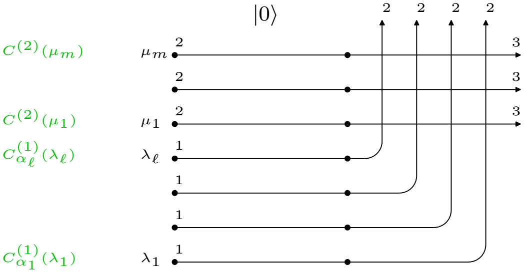
$$\begin{aligned} T^{(2)}_{\delta}(\lambda_{\ell},\dots,\lambda_1|\mu) &= R^{(2)}_{\delta\alpha_{\ell}}(\mu,\lambda_{\ell}) \dots R^{(2)}_{\delta\alpha_1}(\mu,\lambda_1) D^{(1)}_{\delta}(\mu) \\ &= \left(\begin{array}{cc} A^{(2)}(\lambda_{\ell},\dots,\lambda_1|\mu) & C^{(2)}(\lambda_{\ell},\dots,\lambda_1|\mu) \\ B^{(2)}(\lambda_{\ell},\dots,\lambda_1|\mu) & D^{(2)}(\lambda_{\ell},\dots,\lambda_1|\mu) \end{array} \right)_{\delta} \end{aligned}$$

Off-shell Bethe vectors

- Following the nested Bethe Ansatz, one proposes that the states in $\mathcal{H}$

$$|\Psi\rangle = |\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\rangle = C^{(1)}_{\alpha_1}(\lambda_1)\dots C^{(1)}_{\alpha_\ell}(\lambda_\ell)C^{(2)}(\mu_1)\dots C^{(2)}(\mu_m)|0\rangle \otimes |\Uparrow_\alpha\rangle$$

  are eigenvectors of the transfer matrix. We refer to these as *off-shell* Bethe vectors.
- The vector $|\Uparrow_\alpha\rangle = \otimes_{i=1}^{\ell} |\uparrow\rangle_{\alpha_i}$ is needed to fully contract out the vector spaces $V^*_{\alpha_i}$.
- The Bethe vectors admit a convenient graphical representation:

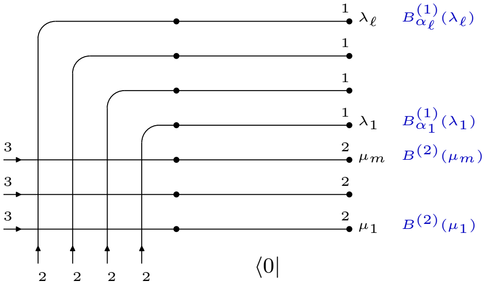


Off-shell dual Bethe vectors

- Similarly, one proposes that the states in $\mathcal{H}^*$

$$\langle\Psi| = \langle\boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell| = \langle\Uparrow_\alpha| \otimes \langle 0| B^{(2)}(\mu_1) \dots B^{(2)}(\mu_m) B^{(1)}_{\alpha_1}(\lambda_1) \dots B^{(1)}_{\alpha_\ell}(\lambda_\ell)$$

  are eigenvectors of the transfer matrix. These are off-shell dual Bethe vectors.
- The vector $\langle\Uparrow_\alpha| = \otimes_{i=1}^{\ell}\langle\uparrow|_{\alpha_i}$ is present to contract out the vector spaces V_{α_i}.
- The dual Bethe vectors have a similar graphical representation:



Bethe equations and on-shell states

- In order to obtain genuine eigenstates of the transfer matrix, one imposes the Bethe equations on the sets $\boldsymbol{\lambda}_\ell$ and $\boldsymbol{\mu}_m$:

$$r_1(\lambda_i) = \frac{a_1(\lambda_i)}{a_2(\lambda_i)} = -\prod_{k=1}^{\ell}\left(\frac{\lambda_k - \lambda_i - 1}{\lambda_k - \lambda_i + 1}\right)\prod_{k=1}^{m}\left(\frac{\mu_k - \lambda_i + 1}{\mu_k - \lambda_i}\right), \qquad 1 \leqslant i \leqslant \ell$$

$$r_3(\mu_j) = \frac{a_3(\mu_j)}{a_2(\mu_j)} = -\prod_{k=1}^{m}\left(\frac{\mu_j - \mu_k - 1}{\mu_j - \mu_k + 1}\right)\prod_{k=1}^{\ell}\left(\frac{\mu_j - \lambda_k + 1}{\mu_j - \lambda_k}\right), \qquad 1 \leqslant j \leqslant m$$

- For the purpose of future calculations, it is useful to introduce the functions

$$\beta_1\left(\nu\middle|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\right) = 1 + r_1(\nu)\prod_{j=1}^{m}\left(\frac{\mu_j - \nu}{\mu_j - \nu + 1}\right)\prod_{i=1}^{\ell}\left(\frac{\lambda_i - \nu + 1}{\lambda_i - \nu - 1}\right)$$

$$\beta_3\left(\nu\middle|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\right) = 1 + r_3(\nu)\prod_{i=1}^{\ell}\left(\frac{\nu - \lambda_i}{\nu - \lambda_i + 1}\right)\prod_{j=1}^{m}\left(\frac{\nu - \mu_j + 1}{\nu - \mu_j - 1}\right)$$

- In terms of these, the Bethe equations are simply

$$\beta_1\left(\lambda_i\middle|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\right) = 0, \quad \forall\ 1 \leqslant i \leqslant \ell, \quad \text{and} \quad \beta_3\left(\mu_j\middle|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\right) = 0, \quad \forall\ 1 \leqslant j \leqslant m.$$

Bethe equations and on-shell states

- Also for later convenience, let us re-normalize the Bethe vectors and the transfer matrix:

$$\|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\rangle\!\rangle = \frac{|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\rangle}{f(\boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell) a_2(\boldsymbol{\lambda}_\ell) a_2(\boldsymbol{\mu}_m)}, \quad \langle\!\langle \boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell\| = \frac{\langle \boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell|}{f(\boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell) a_2(\boldsymbol{\lambda}_\ell) a_2(\boldsymbol{\mu}_m)}$$

$$\mathbb{T}(z) = \sum_{k=1}^{3} \frac{T_{kk}(z)}{a_2(z)}$$

- Assuming that the Bethe equations hold, the Bethe vectors that we have defined become eigenstates of the transfer matrix:

$$\mathbb{T}(z)\|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\rangle\!\rangle = \Lambda\Big(z\Big|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\Big)\|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\rangle\!\rangle, \qquad \langle\!\langle \boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell\|\mathbb{T}(z) = \Lambda\Big(z\Big|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\Big)\langle\!\langle \boldsymbol{\mu}_m, \boldsymbol{\lambda}_\ell\|$$

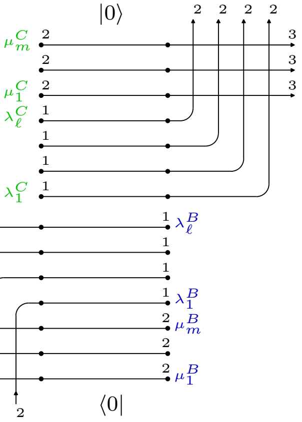
- The eigenvalue, which is the same for both on-shell Bethe vectors and their duals, is given by

$$\Lambda\Big(z\Big|\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m\Big) = r_1(z)\prod_{i=1}^{\ell} f(\lambda_i, z) + \prod_{i=1}^{\ell} f(z, \lambda_i) \prod_{j=1}^{m} f(\mu_j, z) + r_3(z)\prod_{j=1}^{m} f(z, \mu_j)$$

$SU(3)$ scalar product

- The scalar product of the model is simply defined as

$$\mathcal{S}_{\ell,m}(\boldsymbol{\mu}_m^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} | \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}}) = \langle\!\langle \boldsymbol{\mu}_m^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} \| \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}} \rangle\!\rangle$$



- In what follows, we are interested in the case where $\boldsymbol{\lambda}_\ell^{\boldsymbol{B}}, \boldsymbol{\mu}_m^{\boldsymbol{B}}$ are Bethe roots, and $\boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}}$ are free. We refer to this as the *on/off-shell* scalar product.

$SU(3)$ scalar product ($SU(2)$ as a special case)

- The cases $\ell = 0$ and $m = 0$ correspond to an $SU(2)$ scalar product, where the answer was found in determinant form in [Slavnov 89]:

$$\mathcal{S}_{\ell,0}(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}}_\ell | \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell, \emptyset) = \frac{\det\left\{S_j(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda^C_i)\right\}}{\overleftarrow{\Delta}(\boldsymbol{\lambda}^{\boldsymbol{B}})\overrightarrow{\Delta}(\boldsymbol{\lambda}^{\boldsymbol{C}})}, \quad \mathcal{S}_{0,m}(\boldsymbol{\mu}^{\boldsymbol{B}}_m, \emptyset | \emptyset, \boldsymbol{\mu}^{\boldsymbol{C}}_m) = \frac{\det\left\{S'_j(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu^C_i)\right\}}{\overleftarrow{\Delta}(\boldsymbol{\mu}^{\boldsymbol{B}})\overrightarrow{\Delta}(\boldsymbol{\mu}^{\boldsymbol{C}})}$$

- The functions within these determinants are defined to be

$$S_j(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda^C_i) = \frac{1}{\lambda^B_j - \lambda^C_i}\left(r_1(\lambda^C_i) \prod_{k\neq j}^{\ell} (\lambda^B_k - \lambda^C_i + 1) - \prod_{k\neq j}^{\ell} (\lambda^B_k - \lambda^C_i - 1) \right)$$

$$S'_j(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu^C_i) = \frac{1}{\mu^B_j - \mu^C_i}\left(\prod_{k\neq j}^{m} (\mu^B_k - \mu^C_i + 1) - r_3(\mu^C_i) \prod_{k\neq j}^{m} (\mu^B_k - \mu^C_i - 1) \right)$$

- We introduce generalizations of these functions:

$$S_j(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_i^C) =$$

$$\frac{1}{\lambda_j^B-\lambda_i^C}\left(r_1(\lambda_i^C)\prod_{k=1}^{m}\left(\frac{\mu_k^B-\lambda_i^C}{\mu_k^B-\lambda_i^C+1}\right)\prod_{k\neq j}^{\ell}(\lambda_k^B-\lambda_i^C+1)-\prod_{k\neq j}^{\ell}(\lambda_k^B-\lambda_i^C-1)\right)$$

$$S'_j(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_i^C) =$$

$$\frac{1}{\mu_j^B-\mu_i^C}\left(\prod_{k\neq j}^{m}(\mu_k^B-\mu_i^C+1)-r_3(\mu_i^C)\prod_{k=1}^{\ell}\left(\frac{\mu_i^C-\lambda_k^B}{\mu_i^C-\lambda_k^B+1}\right)\prod_{k\neq j}^{m}(\mu_k^B-\mu_i^C-1)\right)$$

Multiple integral expressions [MW 13]

- Define the extended Slavnov-type determinant

$$\mathbb{S}\Big(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}\Big|\boldsymbol{\lambda^C}\Big|\boldsymbol{x}\Big) =$$

$$\frac{\det\left(\begin{array}{ccc|ccc} S_1(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_1^C) & \cdots & S_\ell(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_1^C) & g(x_1,\lambda_1^C) & \cdots & g(x_m,\lambda_1^C) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_1(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_\ell^C) & \cdots & S_\ell(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_\ell^C) & g(x_1,\lambda_\ell^C) & \cdots & g(x_m,\lambda_\ell^C) \\ S_1(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_1^B) & \cdots & S_\ell(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_1^B) & g(x_1,\mu_1^B) & \cdots & g(x_m,\mu_1^B) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_1(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_m^B) & \cdots & S_\ell(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_m^B) & g(x_1,\mu_m^B) & \cdots & g(x_m,\mu_m^B) \end{array}\right)}{\overrightarrow{\Delta}(\boldsymbol{\lambda^B})\overleftarrow{\Delta}(\boldsymbol{\lambda^C}\oplus\boldsymbol{\mu^B})}$$

Multiple integral expressions [MW 13]

- The scalar product of an on-shell dual state $\langle\!\langle \boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} \|$ and an off-shell state $\| \boldsymbol{\lambda}^{\boldsymbol{C}}, \boldsymbol{\mu}^{\boldsymbol{C}} \rangle\!\rangle$ is given by the multiple integral formula

$$\mathcal{S}_{\ell,m}(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} | \boldsymbol{\lambda}^{\boldsymbol{C}}, \boldsymbol{\mu}^{\boldsymbol{C}}) =$$

$$\oint_{\mathcal{X}_m} \frac{dx_m}{2\pi i} \oint_{\mathcal{Y}_m} \frac{dy_m}{2\pi i} \cdots \oint_{\mathcal{X}_1} \frac{dx_1}{2\pi i} \oint_{\mathcal{Y}_1} \frac{dy_1}{2\pi i}\, \mathbb{S}\Big(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} \Big| \boldsymbol{\lambda}^{\boldsymbol{C}} \Big| \boldsymbol{x}\Big) g(\boldsymbol{\mu}^{\boldsymbol{C}}, \boldsymbol{y}) \overrightarrow{\Delta}(\boldsymbol{y}) \times$$
$$\prod_{k=1}^{m} g(x_k, y_k) h(x_k, \boldsymbol{X}_k) h(\boldsymbol{Y}_k, y_k) \left(\frac{\beta_1(x_k | \boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(x_k, \mu^B_k)} - \frac{\beta_3(y_k | \boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(y_k, \mu^B_k)} \right) \frac{g(y_k, \boldsymbol{\mu}^{\boldsymbol{B}}_k)}{g(x_k, \bar{\boldsymbol{\mu}}^{\boldsymbol{B}}_k)}$$

- $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ denote the sets

$$\boldsymbol{X}_k = \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell \oplus \boldsymbol{\mu}^{\boldsymbol{B}}_{k-1} \ominus \boldsymbol{x}_{k-1}, \qquad \boldsymbol{Y}_k = \boldsymbol{\mu}^{\boldsymbol{C}}_m \oplus \boldsymbol{\mu}^{\boldsymbol{B}}_{k-1} \ominus \boldsymbol{y}_{k-1}$$

- The integration contours surround the points

$$\mathcal{X}_k \supset \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell \oplus \boldsymbol{\mu}^{\boldsymbol{B}}_m, \qquad \mathcal{Y}_k \supset \boldsymbol{\mu}^{\boldsymbol{C}}_m$$

Multiple integral expressions [MW 13]

- Define another extended Slavnov determinant:

$$\mathbb{S}'\left(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} \middle| \boldsymbol{\mu}^{\boldsymbol{C}} \middle| \boldsymbol{y}\right) =$$

$$\frac{\det \left(\begin{array}{ccc|ccc} g(y_1,\lambda_1^B) & \cdots & g(y_\ell,\lambda_1^B) & S_1'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_1^B) & \cdots & S_m'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_1^B) \\ \vdots & & \vdots & \vdots & & \vdots \\ g(y_1,\lambda_\ell^B) & \cdots & g(y_\ell,\lambda_\ell^B) & S_1'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_\ell^B) & \cdots & S_m'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\lambda_\ell^B) \\ g(y_1,\mu_1^C) & \cdots & g(y_\ell,\mu_1^C) & S_1'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_1^C) & \cdots & S_m'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_1^C) \\ \vdots & & \vdots & \vdots & & \vdots \\ g(y_1,\mu_m^C) & \cdots & g(y_\ell,\mu_m^C) & S_1'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_m^C) & \cdots & S_m'(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\mu_m^C) \end{array} \right)}{\overleftarrow{\Delta}(\boldsymbol{\mu^B})\overrightarrow{\Delta}(\boldsymbol{\mu^C} \oplus \boldsymbol{\lambda^B})}$$

Multiple integral expressions [MW 13]

- The scalar product of an on-shell dual state $\langle\!\langle \boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} \|$ and an off-shell state $\| \boldsymbol{\lambda}^{\boldsymbol{C}}, \boldsymbol{\mu}^{\boldsymbol{C}} \rangle\!\rangle$ is also given by

$$\mathcal{S}_{\ell,m}(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} | \boldsymbol{\lambda}^{\boldsymbol{C}}, \boldsymbol{\mu}^{\boldsymbol{C}}) =$$

$$\oint_{\mathcal{X}_\ell} \frac{dx_\ell}{2\pi i} \oint_{\mathcal{Y}_\ell} \frac{dy_\ell}{2\pi i} \cdots \oint_{\mathcal{X}_1} \frac{dx_1}{2\pi i} \oint_{\mathcal{Y}_1} \frac{dy_1}{2\pi i}\, \mathbb{S}'\Big(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} \Big| \boldsymbol{\mu}^{\boldsymbol{C}} \Big| \boldsymbol{y}\Big) g(\boldsymbol{x}, \boldsymbol{\lambda}^{\boldsymbol{C}}) \overleftarrow{\Delta}(\boldsymbol{x}) \times$$

$$\prod_{k=1}^{\ell} g(x_k, y_k) h(x_k, \boldsymbol{X}_k) h(\boldsymbol{Y}_k, y_k) \left(\frac{\beta_1(x_k | \boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(x_k, \lambda^B_k)} - \frac{\beta_3(y_k | \boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(y_k, \lambda^B_k)} \right) \frac{g(\boldsymbol{\lambda}^{\boldsymbol{B}}_k, x_k)}{g(\bar{\boldsymbol{\lambda}}^{\boldsymbol{B}}_k, y_k)}$$

- $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ denote the sets

$$\boldsymbol{X}_k = \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell \oplus \boldsymbol{\lambda}^{\boldsymbol{B}}_{k-1} \ominus \boldsymbol{x}_{k-1}, \qquad \boldsymbol{Y}_k = \boldsymbol{\mu}^{\boldsymbol{C}}_m \oplus \boldsymbol{\lambda}^{\boldsymbol{B}}_{k-1} \ominus \boldsymbol{y}_{k-1}$$

- The integration contours surround the points

$$\mathcal{X}_k \supset \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell, \qquad \mathcal{Y}_k \supset \boldsymbol{\mu}^{\boldsymbol{C}}_m \oplus \boldsymbol{\lambda}^{\boldsymbol{B}}_\ell$$

Sum formula for $SU(3)$ off/off-shell scalar product [Reshetikhin 86]

- Reshetikhin discovered a sum formula for the $SU(3)$ scalar product, in the off/off-shell case:

$$f(\boldsymbol{\mu}^B,\boldsymbol{\lambda}^B)f(\boldsymbol{\mu}^C,\boldsymbol{\lambda}^C)\mathcal{S}_{\ell,m}(\boldsymbol{\mu}^B,\boldsymbol{\lambda}^B|\boldsymbol{\lambda}^C,\boldsymbol{\mu}^C) = \sum Z(\boldsymbol{\lambda}^B_{\mathrm{II}},\boldsymbol{\mu}^C_{\mathrm{I}}|\boldsymbol{\lambda}^C_{\mathrm{II}},\boldsymbol{\mu}^B_{\mathrm{I}})Z(\boldsymbol{\lambda}^C_{\mathrm{I}},\boldsymbol{\mu}^B_{\mathrm{II}}|\boldsymbol{\lambda}^B_{\mathrm{I}},\boldsymbol{\mu}^C_{\mathrm{II}})\times$$
$$f(\boldsymbol{\lambda}^C_{\mathrm{I}},\boldsymbol{\lambda}^C_{\mathrm{II}})f(\boldsymbol{\lambda}^B_{\mathrm{II}},\boldsymbol{\lambda}^B_{\mathrm{I}})f(\boldsymbol{\mu}^C_{\mathrm{II}},\boldsymbol{\mu}^C_{\mathrm{I}})f(\boldsymbol{\mu}^B_{\mathrm{I}},\boldsymbol{\mu}^B_{\mathrm{II}})f(\boldsymbol{\mu}^B_{\mathrm{II}},\boldsymbol{\lambda}^B_{\mathrm{II}})f(\boldsymbol{\mu}^C_{\mathrm{I}},\boldsymbol{\lambda}^C_{\mathrm{I}})r_1(\boldsymbol{\lambda}^B_{\mathrm{I}})r_1(\boldsymbol{\lambda}^C_{\mathrm{II}})r_3(\boldsymbol{\mu}^B_{\mathrm{I}})r_3(\boldsymbol{\mu}^C_{\mathrm{II}})$$

- The sum is taken over all partitions of the variables into disjoint subsets:

$$\boldsymbol{\lambda}^C = \boldsymbol{\lambda}^C_{\mathrm{I}} \oplus \boldsymbol{\lambda}^C_{\mathrm{II}}, \quad \boldsymbol{\lambda}^B = \boldsymbol{\lambda}^B_{\mathrm{I}} \oplus \boldsymbol{\lambda}^B_{\mathrm{II}}, \quad \text{such that } |\boldsymbol{\lambda}^B_{\mathrm{I}}| = |\boldsymbol{\lambda}^C_{\mathrm{I}}|, \quad |\boldsymbol{\lambda}^B_{\mathrm{II}}| = |\boldsymbol{\lambda}^C_{\mathrm{II}}|$$
$$\boldsymbol{\mu}^C = \boldsymbol{\mu}^C_{\mathrm{I}} \oplus \boldsymbol{\mu}^C_{\mathrm{II}}, \quad \boldsymbol{\mu}^B = \boldsymbol{\mu}^B_{\mathrm{I}} \oplus \boldsymbol{\mu}^B_{\mathrm{II}}, \quad \text{such that } |\boldsymbol{\mu}^B_{\mathrm{I}}| = |\boldsymbol{\mu}^C_{\mathrm{I}}|, \quad |\boldsymbol{\mu}^B_{\mathrm{II}}| = |\boldsymbol{\mu}^C_{\mathrm{II}}|$$

- This formula generalizes one found in [Korepin 82] [Izergin, Korepin 84] for $SU(2)$-invariant models. By taking either of the cardinalities ℓ or m to be zero, we recover that earlier result.

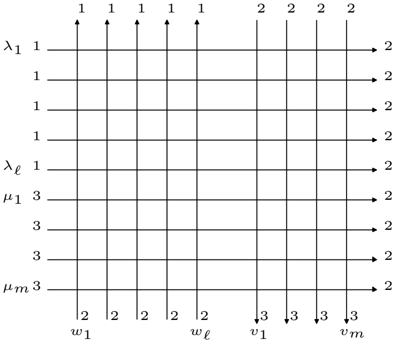
- We can go to the on/off-shell scalar product easily:

$$r_1(\boldsymbol{\lambda}^B_{\mathrm{I}}) \to (-)^{|\boldsymbol{\lambda}^B_{\mathrm{I}}|}\frac{f(\boldsymbol{\lambda}^B_{\mathrm{I}},\boldsymbol{\lambda}^B)}{f(\boldsymbol{\lambda}^B,\boldsymbol{\lambda}^B_{\mathrm{I}})}f(\boldsymbol{\mu}^B,\boldsymbol{\lambda}^B_{\mathrm{I}}), \qquad r_3(\boldsymbol{\mu}^B_{\mathrm{I}}) \to (-)^{|\boldsymbol{\mu}^B_{\mathrm{I}}|}\frac{f(\boldsymbol{\mu}^B,\boldsymbol{\mu}^B_{\mathrm{I}})}{f(\boldsymbol{\mu}^B_{\mathrm{I}},\boldsymbol{\mu}^B)}f(\boldsymbol{\mu}^B_{\mathrm{I}},\boldsymbol{\lambda}^B)$$

Sum formula for $SU(3)$ off/off-shell scalar product [Reshetikhin 86]

- Unfortunately, the function Z is itself a non-trivial object. In [Reshetikhin 86] it was defined as the partition function below:

$Z(\boldsymbol{\lambda}_\ell, \boldsymbol{\mu}_m | \boldsymbol{w}_\ell, \boldsymbol{v}_m) =$



- More recently [MW 12] [Belliard, Pakuliak, Ragoucy, Slavnov 12], it was calculated as a sum over trilinear products of domain wall partition functions.

Properties of the $SU(3)$ off/off-shell scalar product [Reshetikhin 86]

- Despite its complicated form, the off/off-shell scalar product has simple recursive behaviour at *some* of its poles:

$$\lim_{\substack{\mu_m^C \to \mu \\ \mu_m^B \to \mu}} \left\{ (\mu_m^C - \mu_m^B) \mathcal{S}_{\ell,m}(\boldsymbol{\mu}_m^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} | \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}}) \right\} =$$

$$\Big(r_3(\mu_m^C) - r_3(\mu_m^B) \Big) \prod_{j=1}^{m-1} f(\mu, \mu_j^C) f(\mu, \mu_j^B) \mathcal{S}_{\ell,m-1}^{\mathrm{mod}(\mu)} \left(\boldsymbol{\mu}_{m-1}^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} \middle| \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_{m-1}^{\boldsymbol{C}} \right)$$

- The smaller scalar product is modified by scaling its variables r_1, r_3:

$$r_3(y) \mapsto r_3(y) \frac{f(y,\mu)}{f(\mu,y)}, \qquad r_1(x) \mapsto \frac{r_1(x)}{f(\mu,x)}, \qquad \forall \left\{ \begin{array}{l} y \in \boldsymbol{\mu}_{m-1}^{\boldsymbol{B}} \oplus \boldsymbol{\mu}_{m-1}^{\boldsymbol{C}} \\ \\ x \in \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} \oplus \boldsymbol{\lambda}_\ell^{\boldsymbol{C}} \end{array} \right.$$

- Due to symmetry, a similar relation holds for equating any pair $\mu_i^C = \mu_j^B$.
- The scalar product is analytic at the points $\mu_i^C = \lambda_j^B$:

$$\lim_{\mu_m^C \to \lambda_\ell^B} \left\{ (\mu_m^C - \lambda_\ell^B) \mathcal{S}_{\ell,m}(\boldsymbol{\mu}_m^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} | \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}}) \right\} = 0$$

Expectation value of the transfer matrix (acting on on-shell state)

- To derive a recursion relation for the on/off-shell scalar product (*without specializing any of its variables*), we consider the quantity

$$\mathcal{S}_{\ell,m}(z) = \langle\!\langle \boldsymbol{\mu}^{\boldsymbol{B}}_m, \boldsymbol{\lambda}^{\boldsymbol{B}}_\ell \| \mathbb{T}(z) \| \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell, \boldsymbol{\mu}^{\boldsymbol{C}}_m \rangle\!\rangle$$

- Since $\langle\!\langle \boldsymbol{\mu}^{\boldsymbol{B}}_m, \boldsymbol{\lambda}^{\boldsymbol{B}}_\ell \|$ is on-shell, we can easily compute the action of the transfer matrix when it acts left:

$$\mathcal{S}_{\ell,m}(z) = \Lambda(z|\boldsymbol{\lambda}^{\boldsymbol{B}}, \boldsymbol{\mu}^{\boldsymbol{B}}) \langle\!\langle \boldsymbol{\mu}^{\boldsymbol{B}}_m, \boldsymbol{\lambda}^{\boldsymbol{B}}_\ell \| \boldsymbol{\lambda}^{\boldsymbol{C}}_\ell, \boldsymbol{\mu}^{\boldsymbol{C}}_m \rangle\!\rangle$$

- Calculating the residue of $\mathcal{S}_{\ell,m}(z)$ at $z = \mu^B_m$, we obtain

$$\mathrm{res}_{z=\mu^B_m} \Big\{ \mathcal{S}_{\ell,m}(z) \Big\} = \lim_{z \to \mu^B_m} \Big\{ (z - \mu^B_m) \mathcal{S}_{\ell,m}(z) \Big\} =$$

$$\left(r_3(z) \prod_{j=1}^{m-1} f(\mu^B_m, \mu^B_j) - \prod_{j=1}^{m-1} f(\mu^B_j, \mu^B_m) \prod_{i=1}^{\ell} f(\mu^B_m, \lambda^B_i) \right) \mathcal{S}_{\ell,m}(\boldsymbol{\mu}^{\boldsymbol{B}}, \boldsymbol{\lambda}^{\boldsymbol{B}} | \boldsymbol{\lambda}^{\boldsymbol{C}}, \boldsymbol{\mu}^{\boldsymbol{C}})$$

Expectation value of the transfer matrix (acting on off-shell state)

- Now let us calculate the same quantity

$$\mathcal{S}_{\ell,m}(z) = \langle\!\langle \boldsymbol{\mu}_m^{\boldsymbol{B}}, \boldsymbol{\lambda}_\ell^{\boldsymbol{B}} \| \mathbb{T}(z) \| \boldsymbol{\lambda}_\ell^{\boldsymbol{C}}, \boldsymbol{\mu}_m^{\boldsymbol{C}} \rangle\!\rangle$$

  but by acting on the off-shell state, instead.

- To perform this calculation, we use formulae found in [Belliard, Pakuliak, Ragoucy, Slavnov 13]:

$$\begin{aligned}
&\mathbb{T}(z)\|\boldsymbol{\lambda},\boldsymbol{\mu}\rangle\!\rangle = \Lambda(z|\boldsymbol{\lambda},\boldsymbol{\mu})\|\boldsymbol{\lambda},\boldsymbol{\mu}\rangle\!\rangle \\
&+ f(\boldsymbol{\mu},z)\sum_{i=1}^{\ell} g(\lambda_i,z)\left(\prod_{k\neq i}^{\ell} f(\lambda_i,\lambda_k) - \frac{r_1(\lambda_i)}{f(\boldsymbol{\mu},\lambda_i)}\prod_{k\neq i}^{\ell} f(\lambda_k,\lambda_i)\right)\|\widehat{\boldsymbol{\lambda}}_i \oplus z,\boldsymbol{\mu}\rangle\!\rangle \\
&+ f(z,\boldsymbol{\lambda})\sum_{j=1}^{m} g(\mu_j,z)\left(\frac{r_3(\mu_j)}{f(\mu_j,\boldsymbol{\lambda})}\prod_{k\neq j}^{m} f(\mu_j,\mu_k) - \prod_{k\neq j}^{m} f(\mu_k,\mu_j)\right)\|\boldsymbol{\lambda},\widehat{\boldsymbol{\mu}}_j \oplus z\rangle\!\rangle \\
&+ \sum_{i=1}^{\ell}\sum_{j=1}^{m} g(\mu_j,z)g(\mu_j,\lambda_i)\left(\prod_{k\neq i}^{\ell} f(\lambda_i,\lambda_k) - \frac{r_1(\lambda_i)}{f(\boldsymbol{\mu},\lambda_i)}\prod_{k\neq i}^{\ell} f(\lambda_k,\lambda_i)\right)\prod_{k\neq j}^{m} f(\mu_k,\mu_j)\|\widehat{\boldsymbol{\lambda}}_i \oplus z,\widehat{\boldsymbol{\mu}}_j \oplus z\rangle\!\rangle \\
&+ \sum_{i=1}^{\ell}\sum_{j=1}^{m} g(\lambda_i,z)g(\mu_j,\lambda_i)\left(\frac{r_3(\mu_j)}{f(\mu_j,\boldsymbol{\lambda})}\prod_{k\neq j}^{m} f(\mu_j,\mu_k) - \prod_{k\neq j}^{m} f(\mu_k,\mu_j)\right)\prod_{k\neq i}^{\ell} f(\lambda_i,\lambda_k)\|\widehat{\boldsymbol{\lambda}}_i \oplus z,\widehat{\boldsymbol{\mu}}_j \oplus z\rangle\!\rangle
\end{aligned}$$

Expectation value of the transfer matrix (acting on off-shell state)

- We ultimately wish to calculate $\text{res}_{z=\mu^B_m}\{\mathcal{S}_{\ell,m}(z)\}$, and not all of the scalar products resulting from the previous summation have poles at this point.

- The first type of non-zero residue which we will encounter is

$$\lim_{z\to\mu^B_m}\left\{(z-\mu^B_m)\mathcal{S}_{\ell,m}\left(\boldsymbol{\mu}^{\boldsymbol{B}},\boldsymbol{\lambda}^{\boldsymbol{B}}|\boldsymbol{\lambda}^{\boldsymbol{C}},\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\oplus z\right)\right\}$$

$$=\left(r_3(z)-r_3(\mu^B_m)\right)\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\prod_{k=1}^{m-1} f(\mu^B_m,\mu^B_k)\mathcal{S}^{\text{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda}^{\boldsymbol{B}}\middle|\boldsymbol{\lambda}^{\boldsymbol{C}},\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

$$=\left(r_3(z)\prod_{k=1}^{m-1} f(\mu^B_m,\mu^B_k)-f(\mu^B_m,\boldsymbol{\lambda}^{\boldsymbol{B}})\prod_{k=1}^{m-1} f(\mu^B_k,\mu^B_m)\right)\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\mathcal{S}^{\text{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda}^{\boldsymbol{B}}\middle|\boldsymbol{\lambda}^{\boldsymbol{C}},\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

  where the final line follows from the Bethe equations.

- Notice that the factor in blue is common with the expression that we have already found.

Expectation value of the transfer matrix (acting on off-shell state)

- The second type of residue has an analogous form, but the computation is more subtle:

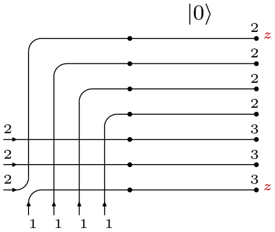
$$\lim_{z\to\mu^B_m}\left\{(z-\mu^B_m)\mathcal{S}_{\ell,m}\left(\boldsymbol{\mu}^{\boldsymbol{B}},\boldsymbol{\lambda}^{\boldsymbol{B}}|\widehat{\boldsymbol{\lambda}}^{\boldsymbol{C}}_i\oplus z,\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\oplus z\right)\right\}$$

$$=\left(r_3(z)-r_3(\mu^B_m)\right)\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\prod_{k=1}^{m-1} f(\mu^B_m,\mu^B_k)\mathcal{S}^{\mathrm{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda}^{\boldsymbol{B}}\middle|\widehat{\boldsymbol{\lambda}}^{\boldsymbol{C}}_i\oplus\mu^B_m,\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

$$=\left(r_3(z)\prod_{k=1}^{m-1} f(\mu^B_m,\mu^B_k)-f(\mu^B_m,\boldsymbol{\lambda}^{\boldsymbol{B}})\prod_{k=1}^{m-1} f(\mu^B_k,\mu^B_m)\right)$$

$$\times\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\mathcal{S}^{\mathrm{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda}^{\boldsymbol{B}}\middle|\widehat{\boldsymbol{\lambda}}^{\boldsymbol{C}}_i\oplus\mu^B_m,\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

- Crucially, $\mathcal{S}_{\ell,m}(\boldsymbol{\mu}^{\boldsymbol{B}},\boldsymbol{\lambda}^{\boldsymbol{B}}|\widehat{\boldsymbol{\lambda}}^{\boldsymbol{C}}_i\oplus z,\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\oplus z)$ does *not* depend on $r_1(z)$:



Recursion relation for on/off-shell scalar product

- We equate the result of acting on the left with the result of acting on the right, and cancel the common factor in blue. We obtain the recursion relation

$$S_{\ell,m}(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\boldsymbol{\lambda^C},\boldsymbol{\mu^C}) =$$

$$-f(\mu^B_m,\boldsymbol{\lambda^C})\sum_{j=1}^{m}\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\prod_{k\neq j}^{m} f(\mu^C_k,\mu^C_j)g(\mu^C_j,\mu^B_m)\beta_3(\mu^C_j|\boldsymbol{\lambda^C},\boldsymbol{\mu^C})S^{\mathrm{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda^B}\middle|\boldsymbol{\lambda^C},\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

$$+\sum_{i=1}^{\ell}\sum_{j=1}^{m} g(\mu^C_j,\lambda^C_i)\prod_{k\neq j}^{m} f(\mu^B_m,\mu^C_k)\prod_{k\neq i}^{\ell} f(\lambda^C_i,\lambda^C_k)\prod_{k\neq j}^{m} f(\mu^C_k,\mu^C_j)$$

$$\times\left(g(\mu^C_j,\mu^B_m)\beta_1(\lambda^C_i|\boldsymbol{\lambda^C},\boldsymbol{\mu^C}) - g(\lambda^C_i,\mu^B_m)\beta_3(\mu^C_j|\boldsymbol{\lambda^C},\boldsymbol{\mu^C})\right)S^{\mathrm{mod}(\mu^B_m)}_{\ell,m-1}\left(\widehat{\boldsymbol{\mu}}^{\boldsymbol{B}}_m,\boldsymbol{\lambda^B}\middle|\widehat{\boldsymbol{\lambda}}^{\boldsymbol{C}}_i\oplus\mu^B_m,\widehat{\boldsymbol{\mu}}^{\boldsymbol{C}}_j\right)$$

- This recursion relation can be conveniently written in terms of contour integrals:

$$\frac{S_{\ell,m}(\boldsymbol{\mu^B},\boldsymbol{\lambda^B}|\boldsymbol{\lambda^C},\boldsymbol{\mu^C})}{f(\boldsymbol{\mu^B},\boldsymbol{\mu^C})} = \oint_{\mathcal{X}}\frac{dx}{2\pi i}\oint_{\mathcal{Y}}\frac{dy}{2\pi i}\frac{S^{\mathrm{mod}(\mu^B_m)}_{\ell,m-1}\left(\boldsymbol{\mu^B}\ominus\mu^B_m,\boldsymbol{\lambda^B}\middle|\boldsymbol{\lambda^C}\oplus\mu^B_m\ominus x,\boldsymbol{\mu^C}\ominus y\right)}{f(\boldsymbol{\mu^B}\ominus\mu^B_m,\boldsymbol{\mu^C}\ominus y)}$$

$$\times g(x,y)g(x,\mu^B_m)g(y,\mu^B_m)f(x,\boldsymbol{\lambda^C})\frac{f(\boldsymbol{\mu^C},y)}{f(\boldsymbol{\mu^B},y)}\left(\frac{\beta_1(x|\boldsymbol{\lambda^C},\boldsymbol{\mu^C})}{g(x,\mu^B_m)}-\frac{\beta_3(y|\boldsymbol{\lambda^C},\boldsymbol{\mu^C})}{g(y,\mu^B_m)}\right)$$

- The integration contours surround only the following poles:

$$\mathcal{X}\supset\boldsymbol{\lambda}^{\boldsymbol{C}}_{\ell}\oplus\mu^B_m,\qquad\mathcal{Y}\supset\boldsymbol{\mu}^{\boldsymbol{C}}_m$$

Solution of recursion relation

- It is straightforward to iterate this recursion relation a further $m-1$ times:

$$\frac{S_{\ell,m}(\boldsymbol{\mu}^{\boldsymbol{B}},\boldsymbol{\lambda}^{\boldsymbol{B}}|\boldsymbol{\lambda}^{\boldsymbol{C}},\boldsymbol{\mu}^{\boldsymbol{C}})}{f(\boldsymbol{\mu}^{\boldsymbol{B}},\boldsymbol{\mu}^{\boldsymbol{C}})} =$$

$$\oint_{\mathcal{X}_1}\frac{dx_1}{2\pi i}\oint_{\mathcal{Y}_1}\frac{dy_1}{2\pi i}\cdots\oint_{\mathcal{X}_m}\frac{dx_m}{2\pi i}\oint_{\mathcal{Y}_m}\frac{dy_m}{2\pi i}\, S^{\mathrm{mod}(\boldsymbol{\mu}^{\boldsymbol{B}})}_{\ell,0}\left(\emptyset,\boldsymbol{\lambda}^{\boldsymbol{B}}\middle|\boldsymbol{\lambda}^{\boldsymbol{C}}\oplus\boldsymbol{\mu}^{\boldsymbol{B}}\ominus\boldsymbol{x},\emptyset\right)\times$$

$$\prod_{k=1}^{m} g(x_k,y_k)g(x_k,\mu^B_k)g(y_k,\mu^B_k)f(x_k,\boldsymbol{X}_k)\frac{f(\boldsymbol{Y}_k,y_k)}{f(\boldsymbol{\mu}^{\boldsymbol{B}},y_k)}\left(\frac{\beta_1(x_k|\boldsymbol{X}_k,\boldsymbol{Y}_k)}{g(x_k,\mu^B_k)}-\frac{\beta_3(y_k|\boldsymbol{X}_k,\boldsymbol{Y}_k)}{g(y_k,\mu^B_k)}\right)$$

- The sets $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ are given by

$$\boldsymbol{X}_k=\boldsymbol{\lambda}^{\boldsymbol{C}}_{\ell}\oplus\boldsymbol{\mu}^{\boldsymbol{B}}_{k-1}\ominus\boldsymbol{x}_{k-1},\qquad \boldsymbol{Y}_k=\boldsymbol{\mu}^{\boldsymbol{C}}_{m}\oplus\boldsymbol{\mu}^{\boldsymbol{B}}_{k-1}\ominus\boldsymbol{y}_{k-1}$$

- The integration contours surround the poles

$$\mathcal{X}_k\supset\boldsymbol{\lambda}^{\boldsymbol{C}}_{\ell}\oplus\boldsymbol{\mu}^{\boldsymbol{B}}_{k}\ominus\boldsymbol{x}_{k-1},\qquad \mathcal{Y}_k\supset\boldsymbol{\mu}^{\boldsymbol{C}}_{m}\ominus\boldsymbol{y}_{k-1}$$

- The base of the recursion is a modified $SU(2)$ on/off-shell scalar product, for which all r_1 variables are rescaled:

$$r_1(z)\mapsto\frac{r_1(z)}{f(\boldsymbol{\mu}^{\boldsymbol{B}},z)},\qquad \forall\ z\in\boldsymbol{\lambda}^{\boldsymbol{B}}\oplus\boldsymbol{\lambda}^{\boldsymbol{C}}\oplus\boldsymbol{\mu}^{\boldsymbol{B}}$$

$SU(2)$ on/off-shell scalar product

- By choosing one of the two cardinalities ℓ or m to be zero, one should recover an $SU(2)$ on/off-shell scalar product.
- The case $m = 0$ clearly reproduces the Slavnov determinant formula. In that case there are no integrals at all, and we trivially obtain

$$\mathcal{S}_{\ell,0}(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \boldsymbol{\lambda}^{\boldsymbol{C}}, \emptyset) = \frac{\det \begin{pmatrix} S_1(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda_1^C) & \cdots & S_\ell(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda_1^C) \\ \vdots & & \vdots \\ S_1(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda_\ell^C) & \cdots & S_\ell(\emptyset, \boldsymbol{\lambda}^{\boldsymbol{B}} | \lambda_\ell^C) \end{pmatrix}}{\overrightarrow{\Delta}(\boldsymbol{\lambda}^{\boldsymbol{B}}) \overleftarrow{\Delta}(\boldsymbol{\lambda}^{\boldsymbol{C}})}$$

- The case $\ell = 0$ is more subtle. In that case the determinant in the integrand becomes

$$\mathbb{S}\Big(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset \Big| \emptyset \Big| \boldsymbol{x}\Big) = \frac{1}{\overleftarrow{\Delta}(\boldsymbol{\mu}^{\boldsymbol{B}})} \det \begin{pmatrix} g(x_1, \mu_1^B) & \cdots & g(x_m, \mu_1^B) \\ \vdots & & \vdots \\ g(x_1, \mu_m^B) & \cdots & g(x_m, \mu_m^B) \end{pmatrix}$$

$SU(2)$ on/off-shell scalar product

- The integration over the contours $\mathcal{X}_j$ is now trivialized. In particular, the contour $\mathcal{X}_j$ surrounds a single pole at $x_j = \mu_j^B$, for all $1 \leqslant j \leqslant m$. Evaluating these integrals explicitly, we are left with

$$\mathcal{S}_{0,m}(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \emptyset, \boldsymbol{\mu}^{\boldsymbol{C}}) =$$
$$\oint_{\mathcal{Y}_m} \frac{dy_m}{2\pi i} \cdots \oint_{\mathcal{Y}_1} \frac{dy_1}{2\pi i} \overleftarrow{\Delta}(\boldsymbol{y}) g(\boldsymbol{\mu}^{\boldsymbol{C}}, \boldsymbol{y}) \prod_{k=1}^{m} h(\mathbf{Y}_k, y_k) \beta_3(y_k | \emptyset, \mathbf{Y}_k) g(y_k, \boldsymbol{\mu}_k^{\boldsymbol{B}})$$

- The sets $\boldsymbol{Y}_k$ are unchanged from before:

$$\boldsymbol{Y}_k = \boldsymbol{\mu}_m^{\boldsymbol{C}} \oplus \boldsymbol{\mu}_{k-1}^{\boldsymbol{B}} \ominus \boldsymbol{y}_{k-1}$$

- This multiple integral evaluates to the Slavnov determinant:

$$\mathcal{S}_{0,m}(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \emptyset, \boldsymbol{\mu}^{\boldsymbol{C}}) = \frac{\det \begin{pmatrix} S_1'(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu_1^C) & \cdots & S_m'(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu_1^C) \\ \vdots & & \vdots \\ S_1'(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu_m^C) & \cdots & S_m'(\boldsymbol{\mu}^{\boldsymbol{B}}, \emptyset | \mu_m^C) \end{pmatrix}}{\overrightarrow{\Delta}(\boldsymbol{\mu}^{\boldsymbol{B}}) \overleftarrow{\Delta}(\boldsymbol{\mu}^{\boldsymbol{C}})}$$

Comments and open questions

- When a single set of Bethe roots tends to infinity, $\boldsymbol{\lambda}^{\boldsymbol{B}}_{\ell} \to \infty$ or $\boldsymbol{\mu}^{\boldsymbol{B}}_{m} \to \infty$, the scalar product factorizes into a product of two determinants [MW 12] [Foda, MW 13]. This result can be easily recovered from the multiple integral expressions.

- In the case where the sets $\boldsymbol{\lambda}^{\boldsymbol{C}}_{\ell}$ and $\boldsymbol{\mu}^{\boldsymbol{C}}_{m}$ are also Bethe roots, we recover the norm-squared. In that case, the scalar product is known as a single determinant [Reshetikhin 86] [Belliard, Pakuliak, Ragoucy, Slavnov 12]. How to obtain these results from the multiple integral expression?

- Can the expression be further simplified? It is tempting to speculate that some of the integrations could be performed explicitly:

$$\oint_{\mathcal{Y}_m} \frac{dy_m}{2\pi i} \cdots \oint_{\mathcal{Y}_1} \frac{dy_1}{2\pi i} \overleftarrow{\Delta}(\boldsymbol{y}) g(\boldsymbol{\mu}^{\boldsymbol{C}}, \boldsymbol{y}) \times$$
$$\prod_{k=1}^{m} g(x_k, y_k) h(\boldsymbol{Y}_k, y_k) \left(\frac{\beta_1(x_k|\boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(x_k, \mu^B_k)} - \frac{\beta_3(y_k|\boldsymbol{X}_k, \boldsymbol{Y}_k)}{g(y_k, \mu^B_k)} \right) g(y_k, \boldsymbol{\mu}^{\boldsymbol{B}}_k) = ?$$

- Are these expressions useful for studying more advanced correlation functions of the $SU(3)$-invariant XXX spin chain or in the study of three-point functions in the $SU(3)$ sector of $\mathcal{N} = 4$ SYM [Foda 12] [Foda, Jiang, Kostov, Serban 13]?