Form factors in GL(3)-invariant integrable models

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in Collaboration with

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Standard problem in quantum models is the calculation of matrix elements of operators (form factors)

$$\mathcal{O}_{\psi,\psi'} = \langle\psi|\widehat{\mathcal{O}}|\psi'\rangle$$

where $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates of the Hamiltonian.

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Suppose that the action of $\widehat{\mathcal{O}}$ to the right or to the left is known

$$\widehat{\mathcal{O}}|\psi'\rangle = |\phi'\rangle \qquad \langle\psi|\widehat{\mathcal{O}} = \langle\phi|$$

Then we reduce the problem to the calculation of the scalar product, where one of the states is the eigenstate of the Hamiltonian.

$$\mathcal{O}_{\psi,\psi'} = \langle\psi|\phi'\rangle \qquad \mathcal{O}_{\psi,\psi'} = \langle\phi|\psi'\rangle$$

In the Algebraic Bethe Ansatz solvable models we usually deal with scalar products of two states (Bethe vectors), which depend on sets of complex numbers

$$\langle\psi(\bar{u})|\psi(\bar{u}')\rangle$$

The set $\bar{u} = \{u_1, \dots, u_n\}$ satisfies Bethe equations. The parameters of the set $\bar{u}' = \{u'_1, \dots, u'_n\}$ are considered as arbitrary complex numbers.

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In the case of models with $GL(2)$-invariant R-matrix we have a compact representation for such scalar products, which was found to be convenient both for analytical and numerical calculations. However, in the case of models with $GL(3)$-invariant R-matrix we are not so lucky.

Conjecture

In the models with $GL(3)$-invariant R-matrix an analogue of the representation (1) does not exist.

$$\langle\psi(\bar{u})|\psi(\bar{u}')\rangle$$

The set $\bar{u}$ consists of the roots of Bethe equations.

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The set $\bar{u}'$ consists of arbitrary complex numbers.

In practice the parameters $\bar{u}'$ always satisfy some restrictions.

In the framework of the Algebraic Bethe Ansatz the most fundamental form factors are the ones of the monodromy matrix entries

$$\langle\psi(\bar{u})|T_{ij}(z)|\psi'(\bar{u}')\rangle$$

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Applying the general method we can act with $T_{ij}(z)$ onto one of the states

$$T_{ij}(z)|\psi'(\bar{u}')\rangle = |\phi(\{z,\bar{u}'\})\rangle = \sum_{\bar{u}''}\alpha(\bar{u}'')\,|\psi(\bar{u}'')\rangle, \qquad \bar{u}''\subset\{z,\bar{u}'\}$$

The total set $\{z,\bar{u}'\}$ does not satisfy Bethe equations. However we can not say that the new state $|\phi(\{z,\bar{u}'\})\rangle$ is parameterized by arbitrary complex numbers, since the parameters $\bar{u}'$ are some roots of Bethe equations.

Calculating form factors of the monodromy matrix entries we deal with scalar products

$$\langle\psi(\bar u)|\psi(\bar u'')\rangle$$

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Result 2012: $\langle\psi(\bar{u})|T_{22}(z)|\psi(\bar{u}')\rangle \sim \det \mathcal{N}^{(22)}$

(S. Belliard, S. Pakuliak, E. Ragoucy, N.S., '12)

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Result 2013: $\langle\psi(\bar{u})|T_{\epsilon,\epsilon'}(z)|\psi'(\bar{u}')\rangle \sim \det \mathcal{N}^{(\epsilon,\epsilon')}$

except the form factor of $T_{13}(z)$ (or $T_{31}(z)$).

Algebraic Bethe Ansatz for $GL(3)$-invariant models

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) & T_{13}(z) \\ T_{21}(z) & T_{22}(z) & T_{23}(z) \\ T_{31}(z) & T_{32}(z) & T_{33}(z) \end{pmatrix}$$

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$GL(3)$-invariant R-matrix

$$R(u,v) = \mathbf{I} + g(u,v)\mathbf{P}, \qquad g(u,v) = \frac{c}{u-v}$$

$$(s_1 s_2)\ R_{12}\ (s_1 s_2)^{-1} = R_{12}, \qquad \forall s \in GL(3)$$

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Other rational functions often appearing in the formulas

$$f(u,v) = 1 + g(u,v)$$

$$h(u,v) = \frac{f(u,v)}{g(u,v)}$$

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$$f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v}$$

$$h(u,v) = \frac{f(u,v)}{g(u,v)} = \frac{u-v+c}{c}$$

Shorthand notations for products

$$T_{\epsilon,\epsilon'}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{\epsilon,\epsilon'}(w_k)$$

$$h(\bar{u}, v_j) = \prod_{u_k \in \bar{u}} h(u_k, v_j)$$

$$f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k)$$

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Special subsets

$$\bar{u}_j = \bar{u} \setminus u_j \qquad\qquad f(\bar{u}_j, u_j) = \prod_{\substack{u_k \in \bar{u} \\ u_k \neq u_j}} f(u_k, u_j)$$

Bethe vectors

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

Algebraic Bethe Ansatz works if there exists a pseudovacuum vector $|0\rangle$ and dual pseudovacuum vector $\langle 0|$

$$T_{jj}(u)|0\rangle = r_j(u)|0\rangle, \qquad T_{jk}(u)|0\rangle = 0, \quad j > k$$

$$\langle 0|T_{jj}(u) = r_j(u)\langle 0|, \qquad \langle 0|T_{jk}(u) = 0, \quad j < k$$

One can set one of $r_j(u)$ equals to 1 without loss of generality. Other $r_j(u)$ remain free functional parameters (generalized model). We set $r_2(u) = 1$.

Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \mathsf{tr}\, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

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The first step is to construct special polynomials in creation operators (T_{12}, T_{13}, T_{23}) applied to the pseudovacuum $|0\rangle$.

- Nested Bethe ansatz

(P. Kulish, N. Reshetikhin, '83)

- Other formulations of nested Bethe ansatz

V. Tarasov, A. Varchenko '95

S. Belliard, S. Khoroshkin, S. Pakuliak, E. Ragoucy '08, '10

Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \mathrm{tr}\, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

$$|\psi_{a,b}(\bar{u};\bar{v})\rangle = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle, \qquad i < j$$

$\bar{u} = u_1, \dots, u_a$

$\bar{v} = v_1, \dots, v_b$

$a, b = 0, 1 \dots$

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$$\bar{v} = v_1,\dots,v_b$$
$$a,b = 0,1\dots$$

Example: $a = b = 1$

$$|\psi_{1,1}(u;v)\rangle = T_{12}(u)T_{23}(v)|0\rangle + g(v,u)T_{13}(u)|0\rangle$$

We say that $|\psi_{a,b}(\bar{u};\bar{v})\rangle$ is a Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ are generic complex numbers.

Bethe vectors

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$$|\psi_{a,b}(\bar{u};\bar{v})\rangle = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle, \qquad i < j$$

We say that $|\psi_{a,b}(\bar{u};\bar{v})\rangle$ is an on-shell Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ satisfy the system of Bethe equations

$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)} f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)} f(v_k,\bar{u})$$

Recall: $\bar{u}_k = \bar{u} \setminus u_k, \qquad f(u_k,\bar{u}_k) = \prod_{\substack{u_s\in\bar{u} \\ u_s\ne u_k}} f(u_k,u_s)$

Dual Bethe vectors

Dual Bethe vectors are special polynomials in annihilation operators $(T_{21},\ T_{31},\ T_{32})$ applied to the dual pseudovacuum $\langle 0|$.

$$\langle\psi_{a,b}(\bar u;\bar v)| = \langle 0|P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big), \qquad i>j \qquad \begin{array}{l}\bar u = u_1,\dots,u_a\\ \bar v = v_1,\dots,v_b\\ a,b=0,1\dots\end{array}$$

Example: $a=b=1$

$$\langle\psi_{1,1}(u;v)| = \langle 0|T_{21}(u)T_{32}(v) + g(v,u)\langle 0|T_{31}(u)$$

We say that $\langle\psi_{a,b}(\bar u;\bar v)|$ is a dual on-shell Bethe vector, if the parameters $\bar u$ and $\bar v$ satisfy the system of Bethe equations

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Transfer matrix eigenvalues

On-shell Bethe vectors are eigenvectors of the transfer matrix $\mathcal{T}(w) = \operatorname{tr} T(w)$.

$$\mathcal{T}(w)|\psi_{a,b}(\bar{u};\bar{v})\rangle = \Lambda(w|\bar{u},\bar{v})\ |\psi_{a,b}(\bar{u};\bar{v})\rangle$$

$$\langle\psi_{a,b}(\bar{u};\bar{v})|\mathcal{T}(w) = \Lambda(w|\bar{u},\bar{v})\ \langle\psi_{a,b}(\bar{u};\bar{v})|$$

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$$\Lambda(w|\bar{u},\bar{v}) = r_1(w)f(\bar{u},w) + f(w,\bar{u})f(\bar{v},w) + r_3(w)f(w,\bar{v})$$

$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)}f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)}f(v_k,\bar{u})$$

Form factors of $T_{\epsilon,\epsilon'}(z)$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$$

Here both $\langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|$ and $|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$ are on-shell Bethe vectors.

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Here both $\langle\psi_{a',b'}(\bar{u}^C;\bar{v}^C)|$ and $|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$ are on-shell Bethe vectors.

$$r_1(u_k^B) = \frac{f(u_k^B,\bar{u}_k^B)}{f(\bar{u}_k^B,u_k^B)} f(\bar{v}^B,u_k^B), \qquad r_3(v_k^B) = \frac{f(\bar{v}_k^B,v_k^B)}{f(v_k^B,\bar{v}_k^B)} f(v_k^B,\bar{u}^B)$$

$$r_1(u_k^C) = \frac{f(u_k^C,\bar{u}_k^C)}{f(\bar{u}_k^C,u_k^C)} f(\bar{v}^C,u_k^C), \qquad r_3(v_k^C) = \frac{f(\bar{v}_k^C,v_k^C)}{f(v_k^C,\bar{v}_k^C)} f(v_k^C,\bar{u}^C)$$

Generically $\{\bar{u}^C,\bar{v}^C\}$ and $\{\bar{u}^B,\bar{v}^B\}$ are different solutions of Bethe equations.

Form factors of $T_{\epsilon,\epsilon'}(z)$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$$

The integers a and b are fixed. Then

$a' = a + \delta_{\epsilon,1} - \delta_{\epsilon',1}$,

$b' = b + \delta_{\epsilon',3} - \delta_{\epsilon,3}$.

The parameter z is an arbitrary complex.

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$$\mathcal{F}^{(\epsilon,\epsilon')}_{a,b}(z) \equiv \mathcal{F}^{(\epsilon,\epsilon')}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \langle\psi_{a',b'}(\bar{u}^C;\bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

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$a' = a + \delta_{\epsilon,1} - \delta_{\epsilon',1}$,
$b' = b + \delta_{\epsilon',3} - \delta_{\epsilon,3}$.
The parameter z is an arbitrary complex.

One can use $\mathcal{F}^{(\epsilon,\epsilon')}_{a,b}(z)$ in order to calculate matrix elements of more complicated operators.

$$T_{\epsilon,\epsilon'}(z)T_{\nu,\nu'}(w) = \sum_{\psi_{a,b}(\bar{u};\bar{v})} T_{\epsilon,\epsilon'}(z)\,\frac{|\psi_{a,b}(\bar{u};\bar{v})\rangle\langle\psi_{a,b}(\bar{u};\bar{v})|}{\|\psi_{a,b}(\bar{u};\bar{v})\|^2}\,T_{\nu,\nu'}(w)$$

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Inverse scattering problem

(N. Kitanine, J.M. Maillet, V. Terras, '99, '00)

In the $SU(3)$-invariant XXX Heisenberg chain

$$E_m^{\epsilon',\epsilon} = \mathcal{T}^{m-1}(0)\, T_{\epsilon,\epsilon'}(0)\, \mathcal{T}^{-m}(0)$$

where $E_m^{\epsilon',\epsilon}$ are elementary units in the site m

$$E_m^{\epsilon',\epsilon} = \mathbf{1} \otimes \dots E^{\epsilon',\epsilon} \dots \otimes \mathbf{1}, \qquad \left(E^{\epsilon',\epsilon}\right)_{jk} = \delta_{j\epsilon'}\delta_{k\epsilon}$$

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$$\langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|\, E_m^{\epsilon',\epsilon}\, |\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle = \frac{\Lambda^{m-1}(0|\bar{u}^C, \bar{v}^C)}{\Lambda^{m}(0|\bar{u}^B, \bar{v}^B)}\, \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(0)$$

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There exist 9 matrix elements $T_{\epsilon,\epsilon'}(z)$, thus there exist 9 form factors. However not all of them are independent due to symmetries of the R-matrix and morphisms of the $RTT = TTR$ relation.

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) & T_{13}(z) \\ T_{21}(z) & T_{22}(z) & T_{23}(z) \\ T_{31}(z) & T_{32}(z) & T_{33}(z) \end{pmatrix}$$

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Simple transforms relate the form factor of $T_{\epsilon,\epsilon'}$ with the ones of $T_{\epsilon',\epsilon}$ and $T_{4-\epsilon',4-\epsilon}$

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Particular case

If $a = 0$ or $b = 0$, then actually we deal with $GL(2)$ case.
Let for definiteness $b = 0$.

$$|\psi_a(\bar{u})\rangle \equiv |\psi_{a,0}(\bar{u};\emptyset)\rangle, \qquad \langle\psi_a(\bar{u})| \equiv \langle\psi_{a,0}(\bar{u};\emptyset)|$$

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$$\langle\psi_a(\bar{u}^C)|\psi_a(\bar{u}^B)\rangle \sim \det_a \left(\frac{\partial\Lambda(w|\bar{u}^B)}{\partial u_k^B}\Big|_{w=u_j^C} \right)$$

if $|\psi_a(\bar{u}^B)\rangle$ is an on-shell Bethe vector

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Form factors are matrix elements of $T_{\epsilon,\epsilon'}(z)$ between two on-shell Bethe vectors

$$\mathcal{F}_a^{(\epsilon,\epsilon')}(z) = \langle\psi_{a'}(\bar{u}^C)|\,T_{\epsilon,\epsilon'}(z)\,|\psi_a(\bar{u}^B)\rangle$$

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Form factors are matrix elements of $T_{\epsilon,\epsilon'}(z)$ between two on-shell Bethe vectors

$$\mathcal{F}_a^{(\epsilon,\epsilon')}(z) = \langle\psi_{a'}(\bar{u}^C)|\,T_{\epsilon,\epsilon'}(z)\,|\psi_a(\bar{u}^B)\rangle$$

We can act with $T_{\epsilon,\epsilon'}(z)$ either to the right or to the left.
As a result we obtain two types of determinant representations for form factors:

$$\mathcal{F}_a^{(\epsilon,\epsilon')}(z) \sim \det\left(\frac{\partial\Lambda(w|\bar{u}^B)}{\partial u_k^B}\right), \qquad\qquad w \in \{z, \bar{u}^C\}$$

$$\mathcal{F}_a^{(\epsilon,\epsilon')}(z) \sim \det\left(\frac{\partial\Lambda(w|\bar{u}^C)}{\partial u_k^C}\right), \qquad\qquad w \in \{z, \bar{u}^B\}$$

Main results

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \langle\psi_{a',b'}(\bar{u}^C;\bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

We consider three cases: $(\epsilon,\epsilon') = (1,1),\ (2,2),\ (1,2)$.

Main results

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \langle\psi_{a',b'}(\bar{u}^C;\bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

We consider three cases: $(\epsilon,\epsilon') = (1,1),\ (2,2),\ (1,2)$.

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon')}$$

The pre-factor $H_{a,b}$ is (ϵ,ϵ')-independent

$$H_{a,b} = h(\bar{u}^B,\bar{u}^B)h(\bar{v}^C,\bar{v}^C)f(\bar{v}^C,\bar{u}^B)f(z,\bar{u}^B)f(\bar{v}^C,z)\ \Delta'_{a'}(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^B)\Delta_{b'}(\bar{v}^C)$$

Recall: $$h(\bar{u}^B,\bar{u}^B) = \prod_{u_k^B\in\bar{u}^B}\prod_{u_j^B\in\bar{u}^B} h(u_k^B,u_j^B), \qquad etc.$$

Main results

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \langle\psi_{a',b'}(\bar{u}^C;\bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

We consider three cases: $(\epsilon,\epsilon') = (1,1),\ (2,2),\ (1,2)$.

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon')}$$

The pre-factor $H_{a,b}$ is (ϵ,ϵ')-independent

$$H_{a,b} = h(\bar{u}^B,\bar{u}^B)h(\bar{v}^C,\bar{v}^C)f(\bar{v}^C,\bar{u}^B)f(z,\bar{u}^B)f(\bar{v}^C,z)\ \Delta'_{a'}(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^B)\Delta_{b'}(\bar{v}^C)$$

$$\Delta_n(\bar{x}) = \prod_{j>k}^{n} g(x_j,x_k), \qquad \Delta'_n(\bar{x}) = \prod_{j<k}^{n} g(x_j,x_k)$$

Form factor of T_{12}

$$\mathcal{F}^{(1,2)}_{a,b}(z) = \langle \psi_{a+1,b}(\bar u^C; \bar v^C)|T_{12}(z)|\psi_{a,b}(\bar u^B; \bar v^B)\rangle$$

$$a' = a+1$$
$$b' = b$$

$$\mathcal{F}^{(1,2)}_{a,b}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \begin{array}{c} \left(\begin{array}{cc} (*) & \dfrac{\partial \Lambda(x_k|\bar u^C, \bar v^C)}{\partial u_j^C} \\ \hdashline (*) & \dfrac{\partial \Lambda(x_k|\bar u^B, \bar v^B)}{\partial v_j^B} \end{array}\right) \begin{array}{l} \left.\vphantom{\dfrac{a}{b}}\right\} a+1 \\ \left.\vphantom{\dfrac{a}{b}}\right\} b \end{array} \end{array}$$

Form factor of T_{12}

$$\mathcal{F}_{a,b}^{(1,2)}(z) = \langle\psi_{a+1,b}(\bar{u}^C;\bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

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$$\mathcal{F}_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \begin{array}{c} \left(\begin{array}{cc} (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} \\ \hdashline (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\dfrac{A}{B}}\right\} a+1 \\ \left.\vphantom{\dfrac{A}{B}}\right\} b \end{array}$$

$$\bar{x} = \{u_1^B,\dots,u_a^B,z,v_1^C,\dots,v_b^C\}$$

Form factor of T_{12}

$$\mathcal{F}^{(1,2)}_{a,b}(z) = \langle \psi_{a+1,b}(\bar{u}^C;\bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

$$a' = a+1$$
$$b' = b$$

$$\mathcal{F}^{(1,2)}_{a,b}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \begin{array}{c} \left(\begin{array}{c|c|c} (*)\dfrac{\partial \Lambda(u_k^B|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} & (*)\dfrac{\partial \Lambda(z|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} & (*)\dfrac{\partial \Lambda(v_k^C|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} \\ \hline (*)\dfrac{\partial \Lambda(u_k^B|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} & (*)\dfrac{\partial \Lambda(z|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} & (*)\dfrac{\partial \Lambda(v_k^C|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} \end{array} \right) \begin{array}{l} \}\, a+1 \\ \}\, b \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\qquad\qquad}_{a} & \underbrace{\qquad\qquad\qquad}_{1} & \underbrace{\qquad\qquad\qquad}_{b} \end{array} \end{array}$$

Form factor of T_{12}

$$
\mathcal{N}^{(1,2)} = \begin{array}{c} \left(\begin{array}{c:c:c} \mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\ \hdashline \mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \end{array} \right) \\ \underbrace{\hphantom{\mathcal{N}^{(u)}(u_k^B, u_j^C)}}_{a} \; \underbrace{\hphantom{\mathcal{N}^{(u)}(z, u_j^C)}}_{1} \; \underbrace{\hphantom{\mathcal{N}^{(u)}(v_k^C, u_j^C)}}_{b} \end{array}
\begin{array}{l} \left.\vphantom{\Big(}\right\} a+1 \\ \left.\vphantom{\Big(}\right\} b \\ \vphantom{\Big(} \end{array}
$$

Form factor of T_{12}

$$\mathcal{N}^{(1,2)} = \begin{array}{c}\left(\begin{array}{c|c|c} \mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\ \hline \mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \end{array}\right) \\ \begin{array}{ccc} \underbrace{\qquad\qquad\qquad}_{a} & \underbrace{\qquad\qquad\qquad}_{1} & \underbrace{\qquad\qquad\qquad}_{b} \end{array}\end{array} \begin{array}{l} \left.\vphantom{\Big(}\right\} a+1 \\ \left.\vphantom{\Big(}\right\} b \\ \\ \end{array}$$

$$\mathcal{N}^{(u)}(x_k, u_j^C) = \frac{c}{f(x_k, \bar{u}^B) f(\bar{v}^C, x_k)} \frac{g(x_k, \bar{u}^B)}{g(x_k, \bar{u}^C)} \cdot \frac{\partial \Lambda(x_k | \bar{u}^C, \bar{v}^C)}{\partial u_j^C}$$

$$\mathcal{N}^{(v)}(x_k, v_j^B) = \frac{-c}{f(x_k, \bar{u}^B) f(\bar{v}^C, x_k)} \frac{g(\bar{v}^C, x_k)}{g(\bar{v}^B, x_k)} \cdot \frac{\partial \Lambda(x_k | \bar{u}^B, \bar{v}^B)}{\partial v_j^B}$$

Form factor of T_{12}

$$\mathcal{F}^{(1,2)}_{a,b}(z) = \langle\psi_{a+1,b}(\bar{u}^C;\bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

$$a' = a+1$$
$$b' = b$$

$$\mathcal{F}^{(1,2)}_{a,b}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \begin{array}{l} \left(\begin{array}{cc} (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^C,\bar{v}^C)}{\partial u^C_j} \\ \hdashline (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^B,\bar{v}^B)}{\partial v^B_j} \end{array}\right) \begin{array}{l} \}\, a+1 \\ \}\, b \end{array} \end{array}$$

$$\bar{x} = \{u^B_1,\ldots,u^B_a,z,v^C_1,\ldots,v^C_b\}$$

Form factors of T_{11} and T_{22}

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon)}(z) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | T_{\epsilon,\epsilon}(z) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$a' = a$$
$$b' = b$$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon)}$$

Form factors of T_{11} and T_{22}

$$\mathcal{F}^{(\epsilon,\epsilon)}_{a,b}(z) = \langle\psi_{a,b}(\bar{u}^C;\bar{v}^C)|T_{\epsilon,\epsilon}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

$a' = a$

$b' = b$

$$\mathcal{F}^{(\epsilon,\epsilon)}_{a,b}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon)}$$

$$\mathcal{N}^{(\epsilon,\epsilon)} = \begin{pmatrix} (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} \\ \multicolumn{2}{c}{\text{additional row}} \\ (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} \end{pmatrix} \begin{array}{l} \}\, a \\ \\ \}\, b \end{array}$$

$$\bar{x} = \{u_1^B,\dots,u_a^B,z,v_1^C,\dots,v_b^C\}$$

Form factors of T_{11} and T_{22}

$$
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{array}{c}\left(\begin{array}{c|c|c}
\mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\
\hline
Y_k^{(\epsilon)} & Y_{a+1}^{(\epsilon)} & Y_{a+1+k}^{(\epsilon)} \\
\hline
\mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B)
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\mathcal{N}^{(u)}}\right\} a \\
\left.\vphantom{Y^{(\epsilon)}}\right\} 1 \\
\left.\vphantom{\mathcal{N}^{(v)}}\right\} b
\end{array} \\
\begin{array}{ccc}
\underbrace{\hphantom{\mathcal{N}^{(u)}(u_k^B, u_j^C)}}_{a} & \underbrace{\hphantom{\mathcal{N}^{(u)}(z, u_j^C)}}_{1} & \underbrace{\hphantom{\mathcal{N}^{(u)}(v_k^C, u_j^C)}}_{b}
\end{array}
\end{array}
$$

Form factors of T_{11} and T_{22}

$$\mathcal{N}^{(\epsilon,\epsilon)} = \begin{array}{cc} \left(\begin{array}{c|c|c} \mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\ \hline Y_k^{(\epsilon)} & Y_{a+1}^{(\epsilon)} & Y_{a+1+k}^{(\epsilon)} \\ \hline \mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \end{array}\right) & \begin{array}{l} \}\, a \\ \}\, 1 \\ \}\, b \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\qquad\qquad}_{a} & \underbrace{\qquad\qquad\qquad}_{1} & \underbrace{\qquad\qquad\qquad}_{b} \end{array} & \end{array}$$

$$Y_k^{(2)} = 1, \qquad k = 1, \ldots, a+b+1$$

Form factors of T_{11} and T_{22}

$$
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{array}{c} \left(\begin{array}{c|c|c} \mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\ \hline Y_k^{(\epsilon)} & Y_{a+1}^{(\epsilon)} & Y_{a+1+k}^{(\epsilon)} \\ \hline \mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \end{array}\right) \begin{array}{l} \}\, a \\ \}\, 1 \\ \}\, b \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\qquad}_{a} & \underbrace{\qquad\qquad}_{1} & \underbrace{\qquad\qquad}_{b} \end{array} \end{array}
$$

$$
Y_k^{(1)} = -1 + \frac{u_k^B}{c}\left(\frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)} - 1\right), \qquad k = 1, \dots, a
$$

$$
Y_{a+1+k}^{(1)} = \frac{v_k^C + c}{c}\left(\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)} - 1\right), \qquad k = 1, \dots, b
$$

Form factors of T_{11} and T_{22}

$$
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{array}{c} \left(\begin{array}{c|c|c}
\mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\
\hline
Y_k^{(\epsilon)} & Y_{a+1}^{(\epsilon)} & Y_{a+1+k}^{(\epsilon)} \\
\hline
\mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B)
\end{array}\right) \begin{array}{l} \}\, a \\ \}\, 1 \\ \}\, b \end{array} \\
\begin{array}{ccc} \underbrace{\qquad\qquad\qquad}_{a} & \underbrace{\qquad\qquad\qquad}_{1} & \underbrace{\qquad\qquad\qquad}_{b} \end{array}
\end{array}
$$

$Y_{a+1}^{(1)}$ is an arbitrary number except the case $\bar{u}^C = \bar{u}^B$ and $\bar{v}^C = \bar{v}^B$:

$$
Y_{a+1}^{(1)} = \frac{r_1(z)\, f(\bar{u}, z)}{f(\bar{v}, z) f(z, \bar{u})}
$$

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}}) \, W_{\mathsf{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

(N. Reshetikhin '86)

The sum is taken over partitions:

$$\bar{u}^B = \{\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}}\} \qquad \bar{v}^B = \{\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}}\} \qquad \#\bar{v}^B_{\mathrm{I}} = \#\bar{v}^C_{\mathrm{I}} = 0, 1, \dots, b$$

$$\bar{u}^C = \{\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}\} \qquad \bar{v}^C = \{\bar{v}^C_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}\} \qquad \#\bar{u}^C_{\mathrm{I}} = \#\bar{u}^B_{\mathrm{I}} = 0, 1, \dots, a$$

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})\, W_{\mathrm{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

Recall: $\quad T_{jj}(u)|0\rangle = r_j(u)|0\rangle$

$$r_1(\bar{u}^C_{\mathrm{II}}) = \prod_{u^C_j \in \bar{u}^C_{\mathrm{II}}} r_1(u^C_j), \qquad etc.$$

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}}) \, W_{\text{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

W_{part} are rational functions (they depend on the R-matrix).

Some properties of W_{part}: N. Reshetikhin '86

Explicit form of W_{part}: M. Wheeler '12

S. Belliard, S. Pakuliak, E. Ragoucy, N.S. '12

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}}) \, W_{\mathsf{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

If r_1 and r_3 are free functional parameters, then for different partitions the corresponding rational functions W_{part} are labeled by functionally independent factors $r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})$. Therefore we have no possibility to take the sum over partitions.

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) \underbrace{r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})\, W_{\text{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}}_{\text{New rational function}}$$

If $\bar{u}^B$ and $\bar{v}^B$ satisfy Bethe equations, then we can express $r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})$ in terms of rational functions. Therefore we can take the sum over partitions of the sets $\bar{u}^B$ and $\bar{v}^B$.

How it was calculated

Scalar product of arbitrary Bethe vectors:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) \underbrace{r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})\, W_{\text{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}}_{\text{New rational function}}$$

In the case of form factors most of the parameters from the sets $\bar{u}^C$ and $\bar{v}^C$ also satisfy Bethe equations. Then the corresponding $r_1(u^C_j)$ and $r_3(v^C_j)$ also can be expressed in terms of rational functions. Therefore we obtain possibilities for further summation.

How it was calculated

All form factors can be presented in the form

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) = r_1(z) \cdot W_1^{(\epsilon,\epsilon')} + W_2^{(\epsilon,\epsilon')} + r_3(z) \cdot W_3^{(\epsilon,\epsilon')}$$

The coefficients $W_k^{(\epsilon,\epsilon')}$ are rational functions given in terms of the sums over partitions. These sums can be reduced to determinants.

Questions

$$\mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\bar{u}^C; \bar{v}^C) | T_{13}(z) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = \textbf{?}$$

Questions

$$\mathcal{F}^{(13)}_{a,b}(z) = \langle\psi_{a+1,b+1}(\bar{u}^C;\bar{v}^C)|T_{13}(z)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle = \textbf{?}$$

The action of $T_{13}(z)$ on Bethe vectors is the simplest:

$$T_{13}(z)|\psi_{a,b}(\bar{u};\bar{v})\rangle = |\psi_{a+1,b+1}(\{z,\bar{u}\};\{z,\bar{v}\})\rangle$$

Questions

$$\mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\bar u^C;\bar v^C)|T_{13}(z)|\psi_{a,b}(\bar u^B;\bar v^B)\rangle = \ ?$$

The action of $T_{13}(z)$ on Bethe vectors is the simplest:

$$T_{13}(z)|\psi_{a,b}(\bar u;\bar v)\rangle = |\psi_{a+1,b+1}(\{z,\bar u\};\{z,\bar v\})\rangle$$

The actions of other operators give non-trivial linear combinations of Bethe vectors, for example,

$$T_{12}(z)|\psi_{a,b}(\bar u;\bar v)\rangle = \alpha\ |\psi_{a+1,b}(\{z,\bar u\};\bar v)\rangle + \sum_i \beta_i\ |\psi_{a+1,b}(\{z,\bar u\};\{z,\bar v_i\})\rangle$$

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One of possible ways to solve the problem is the standard method based on Reshetikhin's representation.

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Another possibility is to use a multiple integral representation for scalar products involving on-shell Bethe vector (M. Wheeler '13).

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How in some particular cases multiple integrals can be calculated explicitly in terms of determinants?

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This is too general object. In practice we usually deal with some particular cases of arbitrary Bethe vectors. Determinant representations for such scalar products may exist.

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$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^C_{\mathrm{I}}) r_1(\bar{u}^B_{\mathrm{I}}) r_3(\bar{v}^B_{\mathrm{II}})\, W_{\mathsf{part}} \begin{pmatrix} \bar{u}^C_{\mathrm{I}}, \bar{u}^B_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}} \\ \bar{v}^C_{\mathrm{I}}, \bar{v}^B_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}, \bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

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We have a possibility to take the sum over partitions of the sets $\bar{v}^C$ and $\bar{v}^B$. However it is highly non-trivial to see that

$$\mathcal{S}_{a,b} = 0 \qquad \text{for} \qquad a < b$$

This fact immediately follows from the explicit form of Bethe vectors. In the $SU(3)$-invariant Heisenberg chain

$$|\psi_{a,b}(\bar{u};\bar{v})\rangle = 0, \qquad \langle\psi_{a,b}(\bar{u};\bar{v})| = 0 \qquad \text{for} \qquad a < b$$

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One can hope to solve the problem of scalar products and form factors in specific models.