Form factors in GL(3)-invariant integrable models

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Standard problem in quantum models is the calculation of matrix elements of operators (form factors)

\[ O_{\psi,\psi'} = \langle \psi | \hat{O} | \psi' \rangle \]

where \( |\psi\rangle \) and \( |\psi'\rangle \) are eigenstates of the Hamiltonian.
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Suppose that the action of \( \hat{O} \) to the right or to the left is known

\[ \hat{O} |\psi'\rangle = |\phi'\rangle \quad \langle \psi | \hat{O} = \langle \phi | \]

Then we reduce the problem to the calculation of the scalar product, where one of the states is the eigenstate of the Hamiltonian.

\[ O_{\psi,\psi'} = \langle \psi | \phi' \rangle \quad O_{\phi,\psi'} = \langle \phi | \psi' \rangle \]
In the Algebraic Bethe Ansatz solvable models we usually deal with scalar products of two states (Bethe vectors), which depend on sets of complex numbers

\[ \langle \psi(\bar{u}) | \psi(\bar{u}') \rangle \]

The set \( \bar{u} = \{u_1, \ldots, u_n\} \) satisfies Bethe equations. The parameters of the set \( \bar{u}' = \{u'_1, \ldots, u'_n\} \) are considered as arbitrary complex numbers.
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In the case of models with \( GL(2) \)-invariant \( R \)-matrix we have a compact representation for such scalar products, which was found to be convenient both for analytical and numerical calculations.
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In the case of models with \( GL(2) \)-invariant \( R \)-matrix we have a compact representation for such scalar products, which was found to be convenient both for analytical and numerical calculations. However, in the case of models with \( GL(3) \)-invariant \( R \)-matrix we are not so lucky.

**Conjecture**

In the models with \( GL(3) \)-invariant \( R \)-matrix an analogue of the representation (1) does not exist.
\[ \langle \psi(\bar{u}) | \psi(\bar{u}') \rangle \]

The set \( \bar{u} \) consists of the roots of Bethe equations. The set \( \bar{u}' \) consists of arbitrary complex numbers.
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The set \(\bar{u}\) consists of the roots of Bethe equations. The set \(\bar{u}'\) consists of arbitrary complex numbers.

In practice the parameters \(\bar{u}'\) always satisfy some restrictions.
In the framework of the Algebraic Bethe Ansatz the most fundamental form factors are the ones of the monodromy matrix entries

$$\langle \psi(\bar{u})|T_{ij}(z)|\psi'(\bar{u}')\rangle$$

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In the framework of the Algebraic Bethe Ansatz the most fundamental form factors are the ones of the monodromy matrix entries

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Here both sets \( \bar{u} \) and \( \bar{u}' \) satisfy Bethe equations.

Applying the general method we can act with \( T_{ij}(z) \) onto one of the states

\[ T_{ij}(z) | \psi'(\bar{u}') \rangle = | \phi(\{z, \bar{u}'\}) \rangle = \sum_{\bar{u}''} \alpha(\bar{u}'') | \psi(\bar{u}'') \rangle, \quad \bar{u}'' \subset \{z, \bar{u}'\} \]

The total set \( \{z, \bar{u}'\} \) does not satisfy Bethe equations. However we can not say that the new state \( | \phi(\{z, \bar{u}'\}) \rangle \) is parameterized by arbitrary complex numbers, since the parameters \( \bar{u}' \) are some roots of Bethe equations.
Calculating form factors of the monodromy matrix entries we deal with scalar products

\[ \langle \psi(\bar{u}) | \psi(\bar{u}'') \rangle \]

where certain restrictions are imposed on both sets \( \bar{u} \) and \( \bar{u}'' \).
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$$\langle \psi(\bar{u})|\psi(\bar{u}'') \rangle$$

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Result 2012:

$$\langle \psi(\bar{u})|T_{22}(z)|\psi(\bar{u}') \rangle \sim \det \mathcal{N}^{(22)}$$

(S. Belliard, S. Pakuliak, E. Ragoucy, N.S., ’12)
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Result 2012: \[ \langle \psi(\bar{u}) | T_{22}(z) | \psi(\bar{u}') \rangle \sim \text{det} \mathcal{N}^{(22)} \]

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Result 2013: \[ \langle \psi(\bar{u}) | T_{\epsilon,\epsilon'}(z) | \psi'(\bar{u}') \rangle \sim \text{det} \mathcal{N}^{(\epsilon,\epsilon')} \]

except the form factor of \( T_{13}(z) \) (or \( T_{31}(z) \)).
Algebraic Bethe Ansatz for $GL(3)$-invariant models

\[ R_{12}(u, v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u, v) \]

\[ T(z) = \begin{pmatrix}
T_{11}(z) & T_{12}(z) & T_{13}(z) \\
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\end{pmatrix} \]
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\end{pmatrix}$$

$GL(3)$-invariant $R$-matrix

$$R(u,v) = I + g(u,v)P, \quad g(u,v) = \frac{c}{u-v}$$

$$(s_1s_2) R_{12} (s_1s_2)^{-1} = R_{12}, \quad \forall s \in GL(3)$$
GL(3)-invariant $R$-matrix

$$R(u - v) = I + g(u, v)P, \quad g(u, v) = \frac{c}{u - v}$$

Other rational functions often appearing in the formulas

$$f(u, v) = 1 + g(u, v)$$

$$h(u, v) = \frac{f(u, v)}{g(u, v)}$$
GL(3)-invariant $R$-matrix

\[ R(u - v) = I + g(u, v)P, \quad g(u, v) = \frac{c}{u - v} \]

Other rational functions often appearing in the formulas

\[ f(u, v) = 1 + g(u, v) = \frac{u - v + c}{u - v} \]

\[ h(u, v) = \frac{f(u, v)}{g(u, v)} = \frac{u - v + c}{c} \]
Shorthand notations for products

\[ T_{\epsilon, \epsilon'}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{\epsilon, \epsilon'}(w_k) \]

\[ h(\bar{u}, v_j) = \prod_{u_k \in \bar{u}} h(u_k, v_j) \]

\[ f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k) \]
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Special subsets

\[ \bar{u}_j = \bar{u} \setminus u_j \]

\[ f(\bar{u}_j, u_j) = \prod_{\substack{u_k \in \bar{u} \\ u_k \neq u_j}} f(u_k, u_j) \]
Bethe vectors

\[ R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v) \]

Algebraic Bethe Ansatz works if there exists a pseudovacuum vector \( |0\rangle \) and dual pseudovacuum vector \( \langle 0| \)

\[ T_{jj}(u)|0\rangle = r_j(u)|0\rangle, \quad T_{jk}(u)|0\rangle = 0, \quad j > k \]

\[ \langle 0|T_{jj}(u) = r_j(u)\langle 0|, \quad \langle 0|T_{jk}(u) = 0, \quad j < k \]

One can set one of \( r_j(u) \) equals to 1 without loss of generality. Other \( r_j(u) \) remain free functional parameters (generalized model). We set \( r_2(u) = 1 \).
Bethe vectors

We look for the eigenvectors of the transfer matrix

\[ \mathcal{T}(w) = \text{tr} T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w) \]
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The first step is to construct special polynomials in creation operators \((T_{12}, T_{13}, T_{23})\) applied to the pseudovacuum \(|0\rangle\).

- Nested Bethe ansatz
  (P. Kulish, N. Reshetikhin, '83)
- Other formulations of nested Bethe ansatz
  V. Tarasov, A. Varchenko '95
  S. Belliard, S. Khoroshkin, S. Pakuliak, E. Ragoucy '08, '10
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We look for the eigenvectors of the transfer matrix

\[ \mathcal{T}(w) = \text{tr} \, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w) \]

\[ |\psi_{a,b}(\vec{u}; \vec{v})\rangle = P(T_{ij}(u_k), T_{ij}(v_k))|0\rangle, \quad i < j \]

\[ \vec{u} = u_1, \ldots, u_a \]
\[ \vec{v} = v_1, \ldots, v_b \]
\[ a, b = 0, 1 \ldots \]
Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \text{tr} \, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

$$|\psi_{a,b}(\bar{u}; \bar{v})\rangle = P\left( T_{ij}(u_k), T_{ij}(v_k) \right)|0\rangle, \quad i < j$$

Example: $a = b = 1$

$$|\psi_{1,1}(u; v)\rangle = T_{12}(u)T_{23}(v)|0\rangle + g(v, u)T_{13}(u)|0\rangle$$

We say that $|\psi_{a,b}(\bar{u}; \bar{v})\rangle$ is a Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ are generic complex numbers.
Bethe vectors

We look for the eigenvectors of the transfer matrix

\[ \mathcal{T}(w) = \text{tr} T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w) \]

\[ |\psi_{a,b}(\bar{u}; \bar{v})\rangle = P\left( T_{ij}(u_k), T_{ij}(v_k) \right) |0\rangle, \quad i < j \]

We say that \(|\psi_{a,b}(\bar{u}; \bar{v})\rangle\) is an on-shell Bethe vector, if the parameters \(\bar{u}\) and \(\bar{v}\) satisfy the system of Bethe equations

\[ r_1(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)} f(\bar{v}, u_k), \quad r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u}) \]

Recall: \(\bar{u}_k = \bar{u} \setminus u_k\), \(f(u_k, \bar{u}_k) = \prod_{\substack{u_s \in \bar{u} \setminus \bar{u}_k \ni u_k}} f(u_k, u_s)\)
Dual Bethe vectors

Dual Bethe vectors are special polynomials in annihilation operators \((T_{21}, T_{31}, T_{32})\) applied to the dual pseudovacuum \(\langle 0|\).

\[
\langle \psi_{a,b}(\bar{u}; \bar{v}) | = \langle 0| P(T_{ij}(u_k), T_{ij}(v_k)), \quad i > j
\]

\[
\bar{u} = u_1, \ldots, u_a \quad \bar{v} = v_1, \ldots, v_b \quad a, b = 0, 1 \ldots
\]

Example: \(a = b = 1\)

\[
\langle \psi_{1,1}(u; v) | = \langle 0| T_{21}(u)T_{32}(v) + g(v, u)\langle 0|T_{31}(u)
\]

We say that \(\langle \psi_{a,b}(\bar{u}; \bar{v}) |\) is a dual on-shell Bethe vector, if the parameters \(\bar{u}\) and \(\bar{v}\) satisfy the system of Bethe equations

\[
r_1(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)} f(\bar{v}, u_k), \quad r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u})
\]
Transfer matrix eigenvalues

On-shell Bethe vectors are eigenvectors of the transfer matrix $\mathcal{T}(w) = \text{tr} T(w)$.

\[
\mathcal{T}(w) |\psi_{a,b}(\bar{u}; \bar{v})\rangle = \Lambda(w |\bar{u}, \bar{v}) |\psi_{a,b}(\bar{u}; \bar{v})\rangle
\]

\[
\langle\psi_{a,b}(\bar{u}; \bar{v}) |\mathcal{T}(w) = \Lambda(w |\bar{u}, \bar{v}) \langle\psi_{a,b}(\bar{u}; \bar{v})|
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$$\Lambda(w|\bar{u}, \bar{v}) = r_1(w) f(\bar{u}, w) + f(w, \bar{u}) f(\bar{v}, w) + r_3(w) f(w, \bar{v})$$

$$r_1(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)} f(\bar{v}, u_k), \quad r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u})$$
Form factors of $T_{\epsilon,\epsilon'}(z)$

$$F_{a,b}^{(\epsilon,\epsilon')}(z) \equiv F_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

Here both $\langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|$ and $|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$ are on-shell Bethe vectors.
Form factors of $T_{\epsilon,\epsilon'}(z)$

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Here both $\langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C) |$ and $| \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$ are on-shell Bethe vectors.

$$r_1(u_k^B) = \frac{f(u_k^B, \bar{u}_k^B)}{f(\bar{u}_k^B, u_k^B)} f(\bar{v}^B, u_k^B), \quad r_3(v_k^B) = \frac{f(\bar{v}_k^B, v_k^B)}{f(v_k^B, \bar{v}_k^B)} f(v_k^B, \bar{u}^B)$$

$$r_1(u_k^C) = \frac{f(u_k^C, \bar{u}_k^C)}{f(\bar{u}_k^C, u_k^C)} f(\bar{v}^C, u_k^C), \quad r_3(v_k^C) = \frac{f(\bar{v}_k^C, v_k^C)}{f(v_k^C, \bar{v}_k^C)} f(v_k^C, \bar{u}^C)$$

Generically $\{\bar{u}^C, \bar{v}^C\}$ and $\{\bar{u}^B, \bar{v}^B\}$ are different solutions of Bethe equations.
Form factors of $T_{\epsilon,\epsilon'}(z)$

$$F_{a,b}^{(\epsilon,\epsilon')}(z) \equiv F_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$$

The integers $a$ and $b$ are fixed. Then

$a' = a + \delta_{\epsilon,1} - \delta_{\epsilon',1}$,

$b' = b + \delta_{\epsilon',3} - \delta_{\epsilon,3}$.

The parameter $z$ is an arbitrary complex.
Form factors of $T_{\epsilon, \epsilon'}(z)$

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The integers $a$ and $b$ are fixed. Then

$$a' = a + \delta_{\epsilon,1} - \delta_{\epsilon',1},$$

$$b' = b + \delta_{\epsilon',3} - \delta_{\epsilon,3}.$$  

The parameter $z$ is an arbitrary complex.

One can use $F_{a,b}^{(\epsilon, \epsilon')}(z)$ in order to calculate matrix elements of more complicated operators.

$$T_{\epsilon, \epsilon'}(z)T_{\nu, \nu'}(w) = \sum_{\psi_{a,b}(\bar{u}; \bar{v})} T_{\epsilon, \epsilon'}(z) \frac{|\psi_{a,b}(\bar{u}; \bar{v})\rangle\langle \psi_{a,b}(\bar{u}; \bar{v})|}{\|\psi_{a,b}(\bar{u}; \bar{v})\|^2} T_{\nu, \nu'}(w)$$
Form factors of $T_{\epsilon,\epsilon'}(z)$

\[ F^{(\epsilon,\epsilon')}_{a,b}(z) \equiv F^{(\epsilon,\epsilon')}_{a,b}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

Inverse scattering problem

(N. Kitanine, J.M. Maillet, V. Terras, ’99, ’00)

In the $SU(3)$-invariant $XXX$ Heisenberg chain

\[ E^{'\epsilon,\epsilon'}_m = T^{m-1}(0) T_{\epsilon,\epsilon'}(0) T^{-m}(0) \]

where $E^{'\epsilon,\epsilon'}_m$ are elementary units in the site $m$

\[ E^{'\epsilon,\epsilon'}_m = 1 \otimes \ldots E^{'\epsilon,\epsilon'} \ldots \otimes 1, \quad \left( E^{'\epsilon,\epsilon'} \right)_{jk} = \delta_{j\epsilon'} \delta_{k\epsilon} \]
Form factors of $T_{\epsilon, \epsilon'}(z)$

$$\mathcal{F}_{a,b}^{(\epsilon, \epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon, \epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon, \epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

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$$\langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)| E_{m,\epsilon}^{\epsilon'} |\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = \frac{\Lambda_{m-1}^{m-1}(0|\bar{u}^C, \bar{v}^C)}{\Lambda_{m}^{m}(0|\bar{u}^B, \bar{v}^B)} \mathcal{F}_{a,b}^{(\epsilon, \epsilon')}(0)$$
Form factors of $T_{\epsilon,\epsilon'}(z)$

$$F_{a,b}^{(\epsilon,\epsilon')}(z) \equiv F_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

There exist 9 matrix elements $T_{\epsilon,\epsilon'}(z)$, thus there exist 9 form factors. However not all of them are independent due to symmetries of the $R$-matrix and morphisms of the $RTT = TTR$ relation.

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) & T_{13}(z) \\ T_{21}(z) & T_{22}(z) & T_{23}(z) \\ T_{31}(z) & T_{32}(z) & T_{33}(z) \end{pmatrix}$$
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Simple transforms relate the form factor of $T_{\epsilon,\epsilon'}$ with the ones of $T_{\epsilon',\epsilon}$ and $T_{4-\epsilon',4-\epsilon}$
Form factors of $T_{\epsilon,\epsilon'}(z)$

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Simple transforms relate the form factor of $T_{\epsilon,\epsilon'}$ with the ones of $T_{\epsilon',\epsilon}$ and $T_{4-\epsilon',4-\epsilon}$.
Particular case

If $a = 0$ or $b = 0$, then actually we deal with $GL(2)$ case. Let for definiteness $b = 0$. 

$$|\psi_a(\bar{u})\rangle \equiv |\psi_{a,0}(\bar{u}; \emptyset)\rangle, \quad \langle \psi_a(\bar{u}) | \equiv \langle \psi_{a,0}(\bar{u}; \emptyset) |$$
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We know a determinant representation for the scalar product of an arbitrary Bethe vector with on-shell Bethe vector.
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\]

We know a determinant representation for the scalar product of an arbitrary Bethe vector with on-shell Bethe vector.

\[
\langle \psi_a(\bar{u}^C)|\psi_a(\bar{u}^B)\rangle \sim \det_a \left( \frac{\partial \Lambda(w|\bar{u}^B)}{\partial u^B_k} \right)_{w=\bar{u}^C_j}
\]

if \( |\psi_a(\bar{u}^B)\rangle \) is an on-shell Bethe vector.
**Particular case**

If $a = 0$ or $b = 0$, then actually we deal with $GL(2)$ case. Let for definiteness $b = 0$.

\[
|\psi_a(\bar{u})\rangle \equiv |\psi_{a,0}(\bar{u}; \emptyset)\rangle, \quad \langle \psi_a(\bar{u})| \equiv \langle \psi_{a,0}(\bar{u}; \emptyset)|
\]

We know a determinant representation for the scalar product of an arbitrary Bethe vector with on-shell Bethe vector.

\[
\langle \psi_a(\bar{u}^C)|\psi_a(\bar{u}^B)\rangle \sim \det_a \left( \frac{\partial \Lambda(w|\bar{u}^C)}{\partial u^C_k} \bigg|_{w=u^B_j} \right)
\]

if $\langle \psi_a(\bar{u}^C)|$ is an on-shell Bethe vector
Particular case

Form factors are matrix elements of $T_{\epsilon,\epsilon'}(z)$ between two on-shell Bethe vectors

$$\mathcal{F}_a^{(\epsilon,\epsilon')}(z) = \langle \psi_{a'}(\bar{u}^C) | T_{\epsilon,\epsilon'}(z) | \psi_a(\bar{u}^B) \rangle$$
**Particular case**

Form factors are matrix elements of $T_{\epsilon,\epsilon'}(z)$ between two on-shell Bethe vectors

$$F^{(\epsilon,\epsilon')}_a(z) = \langle \psi_1(\bar{u}_B^C) | T_{\epsilon,\epsilon'}(z) | \psi_a(\bar{u}_B^B) \rangle$$

We can act with $T_{\epsilon,\epsilon'}(z)$ either to the right or to the left. As a result we obtain two types of determinant representations for form factors:

$$F^{(\epsilon,\epsilon')}_a(z) \sim \det \left( \frac{\partial \Lambda(w|\bar{u}_B^B)}{\partial u_k^B} \right), \quad w \in \{z, \bar{u}_B^C\}$$

$$F^{(\epsilon,\epsilon')}_a(z) \sim \det \left( \frac{\partial \Lambda(w|\bar{u}_C^C)}{\partial u_k^C} \right), \quad w \in \{z, \bar{u}_B^B\}$$
Main results

\[ \mathcal{F}_{a,b}^{(\epsilon,\epsilon')} (z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')} (z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi'_{a',b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon,\epsilon'}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

We consider three cases: \((\epsilon, \epsilon') = (1, 1), (2, 2), (1, 2)\).
Main results

\[ F_a, b(\epsilon, \epsilon')(z) \equiv F_{a, b}(\epsilon, \epsilon')(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a', b'}(\bar{u}^C; \bar{v}^C)|T_{\epsilon, \epsilon'}(z)|\psi_{a, b}(\bar{u}^B; \bar{v}^B) \rangle \]

We consider three cases: \((\epsilon, \epsilon') = (1, 1), (2, 2), (1, 2)\).

\[ F_{a, b}(\epsilon, \epsilon') = H_{a, b} \det_{a + b + 1} N^{(\epsilon, \epsilon')} \]

The pre-factor \(H_{a, b}\) is \((\epsilon, \epsilon')\)-independent

\[ H_{a, b} = h(\bar{u}^B, \bar{u}^B)h(\bar{v}^C, \bar{v}^C)f(\bar{v}^C, \bar{u}^B)f(z, \bar{u}^B)f(\bar{v}^C, z) \Delta_{a'}^{(\epsilon)}(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta_{b'}^{(\epsilon)}(\bar{v}^B)\Delta_b(\bar{v}^C) \]

Recall: \(h(\bar{u}^B, \bar{u}^B) = \prod_{u_k^B \in \bar{u}^B} \prod_{u_j^B \in \bar{u}^B} h(u_k^B, u_j^B), \quad \text{etc.}\)
Main results

\[ F^{(\epsilon, \epsilon')}_{a,b}(z) \equiv F^{(\epsilon, \epsilon')}_{a,b}(z|\vec{u}^C, \vec{v}^C; \vec{u}^B, \vec{v}^B) = \langle \psi'_{a',b'}(\vec{u}^C; \vec{v}^C)|T_{\epsilon, \epsilon'}(z)|\psi_{a,b}(\vec{u}^B; \vec{v}^B) \rangle \]

We consider three cases: \((\epsilon, \epsilon') = (1, 1), (2, 2), (1, 2)\).

\[ F^{(\epsilon, \epsilon')}_{a,b}(z) = H_{a,b} \det_{a+b+1} N^{(\epsilon, \epsilon')} \]

The pre-factor \(H_{a,b}\) is \((\epsilon, \epsilon')\)-independent

\[ H_{a,b} = h(\vec{u}^B, \vec{u}^B)h(\vec{v}^C, \vec{v}^C)f(\vec{v}^C, \vec{u}^B)f(z, \vec{u}^B)f(\vec{v}^C, z) \Delta'_{a'}(\vec{u}^C)\Delta_a(\vec{u}^B)\Delta_b'(\vec{v}^B)\Delta_b(\vec{v}^C) \]

\[ \Delta_n(\vec{x}) = \prod_{j>k} g(x_j, x_k), \quad \Delta'_n(\vec{x}) = \prod_{j<k} g(x_j, x_k) \]
Form factor of $T_{12}$

$$\mathcal{F}_{a,b}^{(1,2)}(z) = \langle \psi_{a+1,b}(\bar{u}^C; \bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$$

$$\mathcal{F}_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \begin{pmatrix} \text{(*) } \frac{\partial \Lambda(x_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} & \text{(*) } \frac{\partial \Lambda(x_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \end{pmatrix} \begin{pmatrix} a + 1 \\ b \end{pmatrix}$$

$$a' = a + 1$$

$$b' = b$$
Form factor of $T_{12}$

\[ \mathcal{F}_{a,b}^{(1,2)}(z) = \langle \psi_{a+1,b}(\bar{u}^C; \bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ a' = a + 1 \]
\[ b' = b \]

\[ \mathcal{F}_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)} \]

\[ \mathcal{N}^{(1,2)} = \left( \begin{array}{c} (\ast) \frac{\partial \Lambda(x_k|\bar{u}^C, \bar{v}^C)}{\partial u^C_j} \\ \vdots \end{array} \right) \right\} a + 1 \]

\[ \bar{x} = \{ u_1^B, \ldots, u_a^B, z, v_1^C, \ldots, v_b^C \} \]
Form factor of $T_{12}$

\[ F_{a,b}^{(1,2)}(z) = \langle \psi_{a+1,b}(\bar{u}^C; \bar{v}^C) | T_{12}(z) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ F_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} N^{(1,2)} \]

\[ N^{(1,2)} = \left\{ \begin{array}{c|c|c}
\frac{\partial \Lambda(u_k^B|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} & \frac{\partial \Lambda(z|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} & \frac{\partial \Lambda(v_k^C|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} \\
(\ast) & (\ast) & (\ast) \\
\frac{\partial \Lambda(u_k^B|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} & \frac{\partial \Lambda(z|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} & \frac{\partial \Lambda(v_k^C|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} \\
(\ast) & (\ast) & (\ast) \\
\hline
a & 1 & b
\end{array} \right\} \]

\[ a' = a + 1 \]
\[ b' = b \]
Form factor of $T_{12}$

$$\mathcal{N}^{(1,2)} = \left( \begin{array}{ccc} \mathcal{N}(u)(u_k^B, u_j^C) & \mathcal{N}(u)(z, u_j^C) & \mathcal{N}(u)(v_k^C, u_j^C) \\ \mathcal{N}(v)(u_k^B, v_j^B) & \mathcal{N}(v)(z, v_j^B) & \mathcal{N}(v)(v_k^C, v_j^B) \end{array} \right) \left\{ \begin{array}{c} a + 1 \\ a \quad 1 \quad b \end{array} \right\}$$
Form factor of $T_{12}$

$$\mathcal{N}^{(1,2)} = \left\{ \begin{array}{ccc} \mathcal{N}^{(u)}(u^B_k, u^C_j) & | & \mathcal{N}^{(u)}(z, u^C_j) & | & \mathcal{N}^{(u)}(v^C_k, u^C_j) \\ \mathcal{N}^{(v)}(u^B_k, v^B_j) & | & \mathcal{N}^{(v)}(z, v^B_j) & | & \mathcal{N}^{(v)}(v^C_k, v^B_j) \end{array} \right\} \left\{ \begin{array}{c} a + 1 \\ a \\ 1 \\ b \end{array} \right\}$$

$$\mathcal{N}^{(u)}(x_k, u^C_j) = \frac{c}{f(x_k, \bar{u}^B) f(\bar{v}^C, x_k) g(x_k, \bar{u}^C)} \cdot \frac{\partial \Lambda(x_k | \bar{u}^C, \bar{v}^C)}{\partial u^C_j}$$

$$\mathcal{N}^{(v)}(x_k, v^B_j) = \frac{-c}{f(x_k, \bar{u}^B) f(\bar{v}^C, x_k) g(\bar{v}^B, x_k)} \cdot \frac{\partial \Lambda(x_k | \bar{u}^B, \bar{v}^B)}{\partial v^B_j}$$
Form factor of $T_{12}$

$$F_{a,b}^{(1,2)}(z) = \langle \psi_{a+1,b}(\bar{u}^C; \bar{v}^C)|T_{12}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$$

$$a' = a + 1$$
$$b' = b$$

$$F_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \left( \begin{array}{c}
(* \quad \frac{\partial \Lambda(x_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} \\
(* \quad \frac{\partial \Lambda(x_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} 
\end{array} \right) \right)_{a+1 \choose b}$$

$$\bar{x} = \{u_1^B, \ldots, u_a^B, z, v_1^C, \ldots, v_b^C\}$$
Form factors of $T_{11}$ and $T_{22}$

\[
\mathcal{F}_{a,b}^{(\epsilon,\epsilon)}(z) = \langle \psi_{a,b}(\overline{u}^C; \overline{v}^C)|T_{\epsilon}(z)|\psi_{a,b}(\overline{u}^B; \overline{v}^B)\rangle
\]

\[
\mathcal{F}_{a,b}^{(\epsilon,\epsilon)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon)}
\]

$a' = a$

$b' = b$
Form factors of $T_{11}$ and $T_{22}$

$$
F_{a,b}^{(\epsilon, \epsilon)}(z) = \langle \psi_{a,b}(\bar{u}^C, \bar{v}^C)| T_{\epsilon, \epsilon}(z) | \psi_{a,b}(\bar{u}^B, \bar{v}^B) \rangle
$$

$$
F_{a,b}^{(\epsilon, \epsilon)}(z) = H_{a,b} \det \mathcal{N}^{(\epsilon, \epsilon)}
$$

$$
\mathcal{N}^{(\epsilon, \epsilon)} = \left\{ \begin{array}{c}
\left( * \right) \frac{\partial \Lambda(x_k | \bar{u}^C, \bar{v}^C)}{\partial u^C_j} \\
\text{additional row} \\
\left( * \right) \frac{\partial \Lambda(x_k | \bar{u}^B, \bar{v}^B)}{\partial v^B_j}
\end{array} \right\}
$$

$$
\bar{x} = \{ u^B_1, \ldots, u^B_a, z, v^C_1, \ldots, v^C_b \}
$$
Form factors of $T_{11}$ and $T_{22}$

\[
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{pmatrix}
\mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\
Y^{(\epsilon)}_k & Y^{(\epsilon)}_{a+1} & Y^{(\epsilon)}_{a+1+k} \\
\mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B)
\end{pmatrix}
\]

\[
\begin{aligned}
&\{a\} \\
&\{1\} \\
&\{b\}
\end{aligned}
\]
Form factors of $T_{11}$ and $T_{22}$

\[
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{pmatrix}
\mathcal{N}^{(u)}(u^B_k, u^C_j) & \mathcal{N}^{(u)}(z, u^C_j) & \mathcal{N}^{(u)}(v^C_k, u^C_j) \\
Y^{(\epsilon)}_k & Y^{(\epsilon)}_{a+1} & Y^{(\epsilon)}_{a+1+k} \\
\mathcal{N}^{(v)}(u^B_k, v^B_j) & \mathcal{N}^{(v)}(z, v^B_j) & \mathcal{N}^{(v)}(v^C_k, v^B_j)
\end{pmatrix}
\]

\[
Y^{(2)}_k = 1, \quad k = 1, \ldots, a + b + 1
\]
Form factors of $T_{11}$ and $T_{22}$

$$
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{pmatrix}
\mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\
\mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \\
\end{pmatrix}
$$

$$
Y^{(1)}_k = -1 + \frac{u_k^B}{c} \left( \frac{f_1(v_k^B, u_k^B)}{f_1(v_k^C, u_k^B)} - 1 \right), \quad k = 1, \ldots, a
$$

$$
Y^{(1)}_{a+1+k} = \frac{v_k^C}{c} + \frac{c}{f_1(v_k^C, \bar{u}_k^C)} \left( \frac{f_1(v_k^C, \bar{u}_k^C)}{f_1(v_k^C, \bar{u}_k^B)} - 1 \right), \quad k = 1, \ldots, b
$$
Form factors of $T_{11}$ and $T_{22}$

\[
\mathcal{N}^{(\epsilon,\epsilon)} = \begin{pmatrix}
\mathcal{N}^{(u)}(u_k^B, u_j^C) & \mathcal{N}^{(u)}(z, u_j^C) & \mathcal{N}^{(u)}(v_k^C, u_j^C) \\
Y_{k}^{(\epsilon)} & Y_{a+1}^{(\epsilon)} & Y_{a+1+k}^{(\epsilon)} \\
\mathcal{N}^{(v)}(u_k^B, v_j^B) & \mathcal{N}^{(v)}(z, v_j^B) & \mathcal{N}^{(v)}(v_k^C, v_j^B) \\
_{a} & _{1} & _{b}
\end{pmatrix}
\]

$Y_{a+1}^{(1)}$ is an arbitrary number except the case $\bar{u}^C = \bar{u}^B$ and $\bar{v}^C = \bar{v}^B$:

\[
Y_{a+1}^{(1)} = \frac{r_1(z) f(\bar{u}, z)}{f(\bar{v}, z) f(z, \bar{u})}
\]
How it was calculated

Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]
How it was calculated

Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}^C_I)r_3(\bar{v}^C_I)r_1(\bar{u}^B_I)r_3(\bar{v}^B_I) W_{\text{part}} \left( \frac{\bar{u}^C_I}{\bar{v}^C_I}, \frac{\bar{u}^B_I}{\bar{v}^B_I}, \frac{\bar{u}^C_{II}}{\bar{v}^C_{II}}, \frac{\bar{u}^B_{II}}{\bar{v}^B_{II}} \right) \]

(N. Reshetikhin ‘86)

The sum is taken over partitions:

\[ \bar{u}^B = \{ \bar{u}^B_I, \bar{u}^B_{II} \} \quad \bar{v}^B = \{ \bar{v}^B_I, \bar{v}^B_{II} \} \quad \# \bar{v}^B_I = \# \bar{v}^C_I = 0, 1, \ldots, b \]

\[ \bar{u}^C = \{ \bar{u}^C_I, \bar{u}^C_{II} \} \quad \bar{v}^C = \{ \bar{v}^C_I, \bar{v}^C_{II} \} \quad \# \bar{u}^C_I = \# \bar{u}^B_I = 0, 1, \ldots, a \]
Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}^C_{II}) r_3(\bar{v}^C_1) r_1(\bar{u}^B_1) r_3(\bar{v}^B_II) W_{\text{part}}(\bar{u}^C_1, \bar{u}^B_1, \bar{u}^C_{II}, \bar{u}^B_{II}, \bar{v}^C_1, \bar{v}^B_1, \bar{v}^C_{II}, \bar{v}^B_{II}) \]

Recall: \[ T_{jj}(u)|0\rangle = r_j(u)|0\rangle \]

\[ r_1(\bar{u}^C_{II}) = \prod_{u^C_j \in \bar{u}^C_{II}} r_1(u^C_j), \quad \text{etc.} \]
How it was calculated

Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_{II}^C)r_3(\bar{v}_I^C)r_1(\bar{u}_I^B)r_3(\bar{v}_{II}^B) W_{\text{part}} \left( \frac{\bar{u}_I^C}{\bar{v}_I^C}, \frac{\bar{u}_I^B}{\bar{v}_I^B}, \frac{\bar{u}_{II}^C}{\bar{v}_{II}^C}, \frac{\bar{u}_{II}^B}{\bar{v}_{II}^B} \right) \]

\( W_{\text{part}} \) are rational functions (they depend on the \( R \)-matrix).

Some properties of \( W_{\text{part}} \): N. Reshetikhin ’86

Explicit form of \( W_{\text{part}} \): M. Wheeler ’12

S. Belliard, S. Pakuliak, E. Ragoucy, N.S. ’12
How it was calculated

Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_I^C)r_3(\bar{v}_I^C)r_1(\bar{u}_I^B)r_3(\bar{v}_I^B) W_{\text{part}} \left( \frac{\bar{u}_I^C, \bar{u}_I^B, \bar{u}_II^C, \bar{u}_II^B}{\bar{v}_I^C, \bar{v}_I^B, \bar{v}_II^C, \bar{v}_II^B} \right) \]

If \( r_1 \) and \( r_3 \) are free functional parameters, then for different partitions the corresponding rational functions \( W_{\text{part}} \) are labeled by functionally independent factors \( r_1(\bar{u}_II^C)r_3(\bar{v}_I^C)r_1(\bar{u}_I^B)r_3(\bar{v}_II^B) \). Therefore we have no possibility to take the sum over partitions.
How it was calculated

Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_I^C)r_3(\bar{v}_I^C)r_1(\bar{u}_I^B)r_3(\bar{v}_I^B) W_{\text{part}} \left( \frac{\bar{u}_I^C}{\bar{v}_I^C}, \frac{\bar{u}_I^B}{\bar{v}_I^B}, \frac{\bar{u}_I^C}{\bar{v}_I^C}, \frac{\bar{u}_I^B}{\bar{v}_I^B} \right) \]

New rational function

If \( \bar{u}^B \) and \( \bar{v}^B \) satisfy Bethe equations, then we can express \( r_1(\bar{u}_I^B)r_3(\bar{v}_I^B) \) in terms of rational functions. Therefore we can take the sum over partitions of the sets \( \bar{u}^B \) and \( \bar{v}^B \).
Scalar product of arbitrary Bethe vectors:

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) \mid \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_\Pi^C)r_3(\bar{v}_I^C)r_1(\bar{u}_I^B)r_3(\bar{v}_\Pi^B)W_{\text{part}}\left(\frac{\bar{u}_1^C}{\bar{v}_1^C}, \frac{\bar{u}_I^B}{\bar{v}_I^B}, \frac{\bar{u}_\Pi^C}{\bar{v}_\Pi^C}, \frac{\bar{u}_\Pi^B}{\bar{v}_\Pi^B}\right) \]

New rational function

In the case of form factors most of the parameters from the sets \( \bar{u}^C \) and \( \bar{v}^C \) also satisfy Bethe equations. Then the corresponding \( r_1(u_j^C) \) and \( r_3(v_j^C) \) also can be expressed in terms of rational functions. Therefore we obtain possibilities for further summation.
How it was calculated

All form factors can be presented in the form

\[ F_{a,b}^{(\epsilon,\epsilon')} (z) = r_1(z) \cdot W_1^{(\epsilon,\epsilon')} + W_2^{(\epsilon,\epsilon')} + r_3(z) \cdot W_3^{(\epsilon,\epsilon')} \]

The coefficients \( W_k^{(\epsilon,\epsilon') \bigr) } \) are rational functions given in terms of the sums over partitions. These sums can be reduced to determinants.
Questions

\[ \mathcal{F}_{a,b}^{(13)}(z) = \langle \psi_{a+1,b+1}(\bar{u}^C; \bar{v}^C)|T_{13}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = ? \]
The action of $T_{13}(z)$ on Bethe vectors is the simplest:

$$T_{13}(z)|\psi_{a,b}(\bar{u};\bar{v})\rangle = |\psi_{a+1,b+1}(\{z,\bar{u}\};\{z,\bar{v}\})\rangle$$
Questions

\[ \mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\bar{u}^C; \bar{v}^C)|T_{13}(z)\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = ? \]

The action of \( T_{13}(z) \) on Bethe vectors is the simplest:

\[ T_{13}(z)|\psi_{a,b}(\bar{u}; \bar{v})\rangle = |\psi_{a+1,b+1}(\{z, \bar{u}\}; \{z, \bar{v}\})\rangle \]

The actions of other operators give non-trivial linear combinations of Bethe vectors, for example,

\[ T_{12}(z)|\psi_{a,b}(\bar{u}; \bar{v})\rangle = \alpha |\psi_{a+1,b}(\{z, \bar{u}\}; \bar{v})\rangle + \sum_i \beta_i |\psi_{a+1,b}(\{z, \bar{u}\}; \{z, \bar{v}_i\})\rangle \]
Questions

\[ \mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\vec{u}^C; \vec{v}^C) | T_{13}(z) | \psi_{a,b}(\vec{u}^B; \vec{v}^B) \rangle = ? \]

The action of \( T_{13}(z) \) on Bethe vectors is the simplest:

\[ T_{13}(z)|\psi_{a,b}(\vec{u}; \vec{v})\rangle = |\psi_{a+1,b+1}(\{z, \vec{u}\}; \{z, \vec{v}\})\rangle \]

One of possible ways to solve the problem is the standard method based on Reshetikhin's representation.

\[ \mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\vec{u}^C; \vec{v}^C) | \psi_{a+1,b+1}(\{z, \vec{u}^B\}; \{z, \vec{v}^B\}) \rangle \]
The action of \( T_{13}(z) \) on Bethe vectors is the simplest:

\[
T_{13}(z) |\psi_{a,b}(\bar{u}; \bar{v})\rangle = |\psi_{a+1,b+1}(\{z, \bar{u}\}; \{z, \bar{v}\})\rangle
\]

Another possibility is to use a multiple integral representation for scalar products involving on-shell Bethe vector (M. Wheeler '13).

\[
\mathcal{F}_{a,b}^{(13)}(z) = \langle \psi_{a+1,b+1}(\bar{u}^C; \bar{v}^C) | \psi_{a+1,b+1}(\{z, \bar{u}^B\}; \{z, \bar{v}^B\}) \rangle
\]
Questions

\[ \mathcal{F}^{(13)}_{a,b}(z) = \langle \psi_{a+1,b+1}(\bar{u}^C; \bar{v}^C)|T_{13}(z)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = ? \]

The action of \( T_{13}(z) \) on Bethe vectors is the simplest:

\[ T_{13}(z)|\psi_{a,b}(\bar{u}; \bar{v})\rangle = |\psi_{a+1,b+1}(\{z, \bar{u}\}; \{z, \bar{v}\})\rangle \]

Another possibility is to use a multiple integral representation for scalar products involving on-shell Bethe vector (M. Wheeler ’13).

How in some particular cases multiple integrals can be calculated explicitly in terms of determinants?
The original idea was to find a determinant representation for the scalar product of an on-shell Bethe vector and arbitrary Bethe vector.
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This is too general object. In practice we usually deal with some particular cases of arbitrary Bethe vectors. Determinant representations for such scalar products may exist.
The generalized model \((r_j(u)\) are free functional parameters) also is too general object. In practice we deal with particular cases of the generalized model. For, instance, in the \(SU(3)\)-invariant Heisenberg chain one has \(r_3(u) = 1\).
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\[
S_{a,b} = \sum r_1(\bar{u}_I^C) r_3(\bar{v}_I^C) r_1(\bar{u}_I^B) r_3(\bar{v}_I^B) W_{\text{part}}\left(\bar{u}_I^C, \bar{u}_I^B, \bar{v}_I^C, \bar{v}_I^B\right)
\]

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We have a possibility to take the sum over partitions of the sets \(\bar{v}_I^C\) and \(\bar{v}_I^B\).
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\]

We have a possibility to take the sum over partitions of the sets \(\bar{v}^C\) and \(\bar{v}^B\). However it is highly non-trivial to see that

\[
S_{a,b} = 0 \quad \text{for} \quad a < b
\]
This fact immediately follows from the explicit form of Bethe vectors. In the $SU(3)$-invariant Heisenberg chain

\[ |\psi_{a,b}(\bar{u}; \bar{v})\rangle = 0, \quad \langle \psi_{a,b}(\bar{u}; \bar{v})| = 0 \quad \text{for} \quad a < b \]

\[ \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C)|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle = 0 \quad \text{for} \quad a < b \]
This fact immediately follows from the explicit form of Bethe vectors. In the $SU(3)$-invariant Heisenberg chain

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One can hope to solve the problem of scalar products and form factors in specific models.