Painlevé functions and conformal blocks

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Example 1 (2D Ising model)

Diagonal two-point correlation function

$$D_N = \langle \sigma(0,0)\sigma(N,N)\rangle_{T<T_c}$$

is a Painlevé VI τ-function [Jimbo, Miwa, '81]:

$$D_N = (1-t)^{\frac{N^2}{2}}\,\tau(t)$$

More explicitly:

$$\left(t(t-1)\sigma''\right)^2 = -2\det\begin{pmatrix} 0 & t\sigma'-\sigma & \sigma'+\frac{2N^2-1}{4} \\ t\sigma'-\sigma & \frac{N^2}{2} & (t-1)\sigma'-\sigma \\ \sigma'+\frac{2N^2-1}{4} & (t-1)\sigma'-\sigma & \frac{N^2}{2} \end{pmatrix}$$

- relation between τ and σ: $\sigma(t) = t(t-1)\dfrac{d}{dt}\ln\tau$
- temperature parameter $t = (\sinh 2\mathcal{K}_x \sinh 2\mathcal{K}_y)^{-2}$, $0<t<1$

Example 2 (exponential fields in SG_{ff} theory)

Two-point correlator

$$D(mr) = \langle e^{i\nu\phi}(0)\, e^{i\nu'\phi}(\mathbf{r})\rangle$$

is a Painlevé III τ-function [SMJ '79, Bernard-Leclair '94]:

$$D\left(2\sqrt{t}\right) = t^{-\frac{(\nu-\nu')^2}{4}} e^{\frac{t}{2}} \tau(t),$$

so that $\sigma(t) = t\frac{d}{dt}\ln\tau$ satisfies

$$\left(t\sigma''\right)^2 = \left(4\left(\sigma'\right)^2 - 1\right)\left(\sigma - t\sigma'\right) + \left(\nu-\nu'\right)^2\left(\sigma' + \frac{1}{2}\right)$$

- long-distance behavior:

$$D(mr) = 1 - \frac{\sin\pi\nu\sin\pi\nu'}{2\pi}\frac{e^{-2mr}}{mr}\left[1 + o\left(1\right)\right]$$

- short-distance behavior:

$$D(mr) = \mathcal{A}(\nu,\nu')\,(mr)^{2\nu\nu'}\left[1 + \text{CPT corrections}\right]$$

Painlevé equations:

- classification of 2nd order ODEs without movable critical points
- non-autonomous hamiltonian systems
- confluence cascade

$$PVI \to PV \nearrow PIV \searrow PII \to PI, \quad PV \searrow PIII \nearrow PII$$

Solutions:

- classical special functions

$${}_2F_1 \to Whittaker \nearrow Hermite \searrow Airy \to \varnothing, \quad Whittaker \searrow Bessel \nearrow Airy$$

- elliptic for PVI
- algebraic
- transcendental (almost all solutions!)

Painlevé VI:

Standard form:

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)\frac{dw}{dt} +$$
$$+ \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left((\theta_\infty - 1/2)^2 - \frac{\theta_0^2 t}{w^2} + \frac{\theta_1^2(t-1)}{(w-1)^2} + \frac{(1/4-\theta_t^2)t(t-1)}{(w-t)^2}\right)$$

Sigma form:

$$-\frac{1}{2}\Big(t(t-1)\sigma''\Big)^2 =$$
$$= \det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

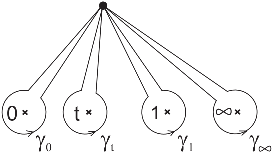
- 3 critical points $t = 0, 1, \infty$
- 4 parameters $\boldsymbol{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$

Painlevé VI and isomonodromy

PVI describes monodromy preserving deformations of rank 2 linear systems on $\mathbb{P}^1$ with 4 regular singular points $0, t, 1, \infty$:

$$\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}$$

- matrices $\mathcal{A}_\nu$ are 2×2, traceless, with eigenvalues $\pm\theta_\nu$
- $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 \overset{def}{=} -\mathcal{A}_\infty = \operatorname{diag}\{-\theta_\infty, \theta_\infty\}$
- 3 monodromy matrices $\mathcal{M}_{0,t,1} \in G = SL(2,\mathbb{C})$ (note $\mathcal{M}_\infty\mathcal{M}_1\mathcal{M}_t\mathcal{M}_0 = \mathbf{1}$)
- monodromy manifold $\mathcal{M} = G^3/G$, $\dim\mathcal{M} = 6$



Painlevé VI and isomonodromy (continued)

Schlesinger equations:

$$\frac{d\mathcal{A}_0}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_0]}{t}, \qquad \frac{d\mathcal{A}_1}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_1]}{t-1}$$

- Lax form $\Rightarrow$ $\theta_{0,t,1,\infty}$ are conserved
- remains 2 degrees of freedom (recall that $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 = -\mathcal{A}_\infty$)
- $\left(\frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}\right)_{12} = \frac{k(t)(z-w(t))}{z(z-t)(z-1)}$ $\Rightarrow$ standard form of PVI for $w(t)$

Derivation of σPVI [Hitchin, '97]:

- define
$$f = \operatorname{tr} \mathcal{A}_0\mathcal{A}_t, \quad g = \operatorname{tr} \mathcal{A}_1\mathcal{A}_t, \quad h = \operatorname{tr} \mathcal{A}_0[\mathcal{A}_t, \mathcal{A}_1]$$
- then for $\sigma = (t-1)f + tg$ we find $\sigma' = f + g$ and $t(t-1)\sigma'' = -h$
- but for any 2×2 traceless $\mathcal{A}_{0,t,1}$
$$(\operatorname{tr} \mathcal{A}_0[\mathcal{A}_t, \mathcal{A}_1])^2 = -2\det \begin{pmatrix} \operatorname{tr} \mathcal{A}_0^2 & \operatorname{tr} \mathcal{A}_0\mathcal{A}_t & \operatorname{tr} \mathcal{A}_0\mathcal{A}_1 \\ \operatorname{tr} \mathcal{A}_t\mathcal{A}_0 & \operatorname{tr} \mathcal{A}_t^2 & \operatorname{tr} \mathcal{A}_t\mathcal{A}_1 \\ \operatorname{tr} \mathcal{A}_1\mathcal{A}_0 & \operatorname{tr} \mathcal{A}_1\mathcal{A}_t & \operatorname{tr} \mathcal{A}_1^2 \end{pmatrix}$$
- and from $\operatorname{tr} \mathcal{A}_\infty^2 = \operatorname{tr}(\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1)^2$ we can find $\operatorname{tr} \mathcal{A}_0\mathcal{A}_1$

Painlevé VI and isomonodromy (continued)

Sigma Painlevé VI:

$$\left(t(t-1)\sigma''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

- to any solution corresponds (the conjugacy class of) a triple $(\mathcal{M}_0, \mathcal{M}_t, \mathcal{M}_1)$
- $p_\nu = 2\cos 2\pi\theta_\nu = \mathrm{tr}\,\mathcal{M}_\nu$ (with $\nu = 0, t, 1, \infty$) give four PVI parameters
- remaining two coordinates $\Rightarrow$ integration constants
- introduce $p_{\mu\nu} = 2\cos 2\pi\sigma_{\mu\nu} = \mathrm{tr}\,\mathcal{M}_\mu\mathcal{M}_\nu$, then [Jimbo, '82]

$$p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - \omega_{0t}p_{0t} - \omega_{1t}p_{1t} - \omega_{01}p_{01} + \omega_4 - 4 = 0, \qquad (1)$$

where $\omega_4 = p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0p_tp_1p_\infty$ and

$$\omega_{0t} = p_0p_t + p_1p_\infty, \quad \omega_{1t} = p_1p_t + p_0p_\infty, \quad \omega_{01} = p_0p_1 + p_tp_\infty$$

The triple $\boldsymbol{\sigma}$ satisfying (1) can be interpreted as a pair of PVI integration constants. Our task is: given $\boldsymbol{\sigma}$, to obtain the corresponding solution.

Jimbo's formula ['82]

- expresses the asymptotics of $\tau(t)$ as $t \to 0$, 1, or ∞ in terms of monodromy
- e.g. for $t \to 0$, denote $\sigma = \sigma_{0t}$ and choose $0 < |\mathrm{Re}\,\sigma| < \frac{1}{2}$
- also denote $\Delta_\nu = \theta_\nu^2$ $(\nu = 0, t, 1, \infty)$ and $\Delta_\sigma = \sigma^2$; then

$$\tau(t) = \mathrm{const} \cdot \left(t^{\Delta_\sigma - \Delta_0 - \Delta_t} + C_{\pm 1} t^{\Delta_\sigma \pm 1 - \Delta_0 - \Delta_t} + \text{ smaller terms } \right),$$

with

$$C_{\pm 1} = \frac{\Gamma^2(1 \mp 2\sigma)}{\Gamma^2(1 \pm 2\sigma)} \prod_{\epsilon = \pm} \frac{\Gamma(1 + \epsilon\theta_0 + \theta_t \pm \sigma)\Gamma(1 + \epsilon\theta_\infty + \theta_1 \pm \sigma)}{\Gamma(1 + \epsilon\theta_0 + \theta_t \mp \sigma)\Gamma(1 + \epsilon\theta_\infty + \theta_1 \mp \sigma)} \times$$
$$\times \frac{\left(\theta_0^2 - (\theta_t \mp \sigma)^2\right)\left(\theta_\infty^2 - (\theta_1 \mp \sigma)^2\right)}{4\sigma^2 (1 \pm 2\sigma)^2} (-s_{0t})^{\pm 1},$$

and

$$s_{0t}^{\pm 1} \left(\cos 2\pi(\theta_t \mp \sigma) - \cos 2\pi\theta_0\right)\left(\cos 2\pi(\theta_1 \mp \sigma) - \cos 2\pi\theta_\infty\right) =$$
$$= \left(\cos 2\pi\theta_t \cos 2\pi\theta_1 + \cos 2\pi\theta_0 \cos 2\pi\theta_\infty \pm i \sin 2\pi\sigma \cos 2\pi\sigma_{01}\right) -$$
$$- \left(\cos 2\pi\theta_0 \cos 2\pi\theta_1 + \cos 2\pi\theta_t \cos 2\pi\theta_\infty \mp i \sin 2\pi\sigma \cos 2\pi\sigma_{1t}\right) e^{\pm 2\pi i \sigma}.$$

- higher-order corrections can be determined recursively from σPVI (in principle)

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta}, \sigma) t + \qquad \ldots\Big)$$

$$+ C_{\pm 1}\; t^{\Delta_{\sigma \pm 1} - \Delta_0 - \Delta_t}$$

with

$$\mathcal{B}_1(\boldsymbol{\theta}, \sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\Big) +$$

$$+ C_{\pm 1}\, t^{\Delta_{\sigma\pm1} - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)t + \ldots\Big) + C_{\pm 2} t^{\Delta_{\sigma\pm2} - \Delta_0 - \Delta_t}\Big(1 + \ldots\Big)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$

$$+ \frac{\Big[(1 + 2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\Big]\Big[(1 + 2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\Big]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2},$$

$$\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t} \Big(1 + \mathcal{B}_1(\boldsymbol{\theta}, \sigma) t + \mathcal{B}_2(\boldsymbol{\theta}, \sigma) t^2 \ldots \Big) +$$
$$+ C_{\pm 1}\, t^{\Delta_{\sigma \pm 1} - \Delta_0 - \Delta_t} \Big(1 + \mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta}, \sigma) t + \ldots \Big) + C_{\pm 2} t^{\Delta_{\sigma \pm 2} - \Delta_0 - \Delta_t} \Big(1 + \ldots \Big)$$

with

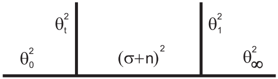
$$\mathcal{B}_1(\boldsymbol{\theta}, \sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta}, \sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$
$$+ \frac{\left[(1 + 2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\right]\left[(1 + 2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\right]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2},$$

$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta}, \sigma) = \mathcal{B}_1(\boldsymbol{\theta}, \sigma \pm 1).$$

Observation. PVI tau function is a linear combination of $c = 1$ conformal blocks:

$$\tau(t) = \sum_{n \in \mathbb{Z}} C_n t^{\Delta_{\sigma + n} - \Delta_0 - \Delta_t} \mathcal{B}\left(\boldsymbol{\theta}, \sigma + n, t\right)$$

Graphical representation of $\mathcal{B}\left(\boldsymbol{\theta}, \sigma + n, t\right)$: 

Higher order corrections (continued)

$$
\begin{aligned}
\mathcal{B}_3(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma-\Delta_0+\Delta_t+2)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)}{24\Delta_\sigma(4\Delta_\sigma-1)^2(\Delta_\sigma-1)^2}\times \\
\times\Big\{(8\Delta_\sigma^2-5\Delta_\sigma+3)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1) \\
-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+2\Delta_1) \\
-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_0+2\Delta_t) \\
+8(6\Delta_\sigma^3+11\Delta_\sigma^2-6\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)\Big\} \\
+\frac{1}{6\Delta_\sigma(\Delta_\sigma-1)^2}\Big\{(\Delta_\sigma^2+3\Delta_\sigma+2)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+3\Delta_1) \\
+(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2) \\
+(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_0+\Delta_t+2) \\
-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2) \\
-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+2)\Big\}
\end{aligned}
$$

- more terms can be checked using computer algebra

General isomonodromy problem

Rank N linear system with n regular singular points $a_1,\ldots,a_n$ on $\mathbb{P}^1$:

$$\partial_z\Phi=\mathcal{A}(z)\Phi,\qquad \mathcal{A}(z)=\sum_{\nu=1}^{n}\frac{\mathcal{A}_\nu}{z-a_\nu}$$

- normalization $\Phi(z_0)=\mathbf{1}_N$
- no singularity at $\infty\Rightarrow\sum_{\nu=1}^{n}\mathcal{A}_\nu=0$
- $\mathcal{A}_\nu$'s assumed to be diagonalizable: $\mathcal{A}_\nu=\mathcal{G}_\nu\mathcal{T}_\nu\mathcal{G}_\nu^{-1}$ with some $\mathcal{T}_\nu=\operatorname{diag}\left\{\lambda_{\nu,1},\ldots,\lambda_{\nu,N}\right\}$
- introducing $\mathcal{J}(z)=\Phi^{-1}\partial_z\Phi=\Phi^{-1}\mathcal{A}(z)\Phi$, expand $\Phi(z)$ around $z=z_0$:

$$\Phi(z\to z_0)=\mathbf{1}_N+\mathcal{J}(z_0)(z-z_0)+\left(\mathcal{J}^2(z_0)+\partial\mathcal{J}(z_0)\right)\frac{(z-z_0)^2}{2}+\ldots$$

- expansions near singular points:

$$\Phi(z\to a_\nu)=\mathcal{G}_\nu(z)\,(z-a_\nu)^{\mathcal{T}_\nu}\,\mathcal{C}_\nu$$

  - $\mathcal{G}_\nu(z)$ is holomorphic and invertible in a neighborhood of $z=a_\nu$, and satisfies $\mathcal{G}_\nu(a_\nu)=\mathcal{G}_\nu$
  - $\mathcal{C}_\nu$ are connection matrices; monodromy matrices $\mathcal{M}_\nu=\mathcal{C}_\nu^{-1}e^{2\pi i\mathcal{T}_\nu}\mathcal{C}_\nu$

General isomonodromy problem (continued)

Deformation equations:

$$\partial_{a_\nu}\Phi = -\frac{z_0 - z}{z_0 - a_\nu}\,\frac{\mathcal{A}_\nu}{z - a_\nu}\,\Phi,$$
$$\partial_{z_0}\Phi = -\mathcal{A}(z_0)\,\Phi.$$

- $\mathcal{J}(z)$ remains invariant under isomonodromic variation of z_0 !

Schlesinger equations:

$$\partial_{a_\mu}\mathcal{A}_\nu = \frac{z_0 - a_\nu}{z_0 - a_\mu}\,\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \mu \neq \nu,$$

$$\partial_{a_\nu}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \partial_{z_0}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{z_0 - a_\mu}.$$

Tau function:

$$d\ln\tau = \sum_{\mu<\nu}\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu\, d\ln(a_\mu - a_\nu).$$

- τ does not depend on z_0 thanks to

$$\partial_{a_\mu}\ln\tau = \sum_{\nu\neq\mu}\frac{\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu}{a_\mu - a_\nu} = \frac{1}{2}\operatorname{res}_{z=a_\mu}\operatorname{tr}\mathcal{J}^2(z).$$

Global conformal symmetry

How does τ trasform under $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$?

Example (three points): $d \ln \tau$ can be explicitly integrated to

$$\tau(a_1, a_2, a_3) = \text{const} \cdot (a_1 - a_2)^{\Delta_3 - \Delta_2 - \Delta_1} (a_1 - a_3)^{\Delta_2 - \Delta_1 - \Delta_3} (a_2 - a_3)^{\Delta_1 - \Delta_2 - \Delta_3},$$

with $\Delta_\nu = \frac{1}{2} \operatorname{tr} \mathcal{A}_\nu^2$ and $\nu = 1, 2, 3$. Expression for 3-point function of quasiprimary fields with dimensions $\Delta_{1,2,3}$ in 2D CFT !

Proposition: One has

$$\tau(f(a)) = \prod_{\nu=1}^{n} \left[f'(a_\nu) \right]^{-\Delta_\nu} \tau(a)$$

■ It suffices to consider infinitesimal transformations generated by $\left(A + Bz + Cz^2\right) \partial_z$
$\Rightarrow$ check three differential constraints

$$\sum_\nu \partial_{a_\nu} \ln \tau = 0,$$

$$\sum_\nu \left(a_\nu \partial_{a_\nu} \ln \tau + \Delta_\nu\right) = 0,$$

$$\sum_\nu \left(a_\nu^2 \partial_{a_\nu} \ln \tau + 2\Delta_\nu a_\nu\right) = 0.$$

□

Ansatz for Φ

Fundamental matrix solution Φ is completely fixed by its monodromy, normalization and singular behaviour (choice of logarithm branches $\mathcal{L}_\nu = \mathcal{C}_\nu^{-1}\mathcal{T}_\nu\mathcal{C}_\nu = \frac{1}{2\pi i}\ln \mathcal{M}_\nu$).

Starting point (cf [Sato, Miwa, Jimbo, '79]):

$$\Phi_{jk}(z) = (z - z_0)^{2\Delta}\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}, \qquad j,k = 1,\ldots,N.$$

Assumptions:

- $\{\mathcal{O}_{\mathcal{L}_\nu}\}$, $\{\bar{\varphi}_j\}$, $\{\varphi_k\}$ are primary fields in a 2D CFT
- OPEs of $\bar{\varphi}$'s with φ's contain $\mathbf{1}$ $\Rightarrow$ same dimensions Δ
- normalization
$$\bar{\varphi}_j(z_0)\varphi_k(z) \sim (z - z_0)^{-2\Delta}\,\delta_{jk}.$$
- dimensions of all other primaries in this OPE are strictly positive integers
- complete OPEs of monodromy fields with auxiliary ones:

$$\mathcal{O}_{\mathcal{L}_\nu}(a_\nu)\varphi_k(z) = \sum_{j=1}^{n}\left((z-a_\nu)^{\mathcal{L}_\nu}\right)_{jk}\sum_{\ell=0}^{\infty}\mathcal{O}_{\mathcal{L}_\nu,j,\ell}(a_\nu)\,(z-a_\nu)^{\ell},$$

If one finds a set of fields with all mentioned properties, the correlator ratio will automatically give Φ.

Tau function

Compute two more orders in the OPE $\bar{\varphi}_j(z_0)\varphi_k(z)$:

$$\bar{\varphi}_j(z_0)\varphi_k(z) = (z-z_0)^{-2\Delta}\Big[\delta_{jk} + J_{jk}(z_0)\,(z-z_0) + \\ +\Big(\frac{4\Delta}{c}\,T(z_0)\delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0)\Big)\frac{(z-z_0)^2}{2} + O\left((z-z_0)^3\right)\Big].$$

- 1st order: no descendants of 1, new primary J
- 2nd order: level 2 descendant of **1**, level 1 descendant of J, new primary S

Comparing with

$$\Phi(z\to z_0) = \mathbf{1}_N + \mathcal{J}\,(z_0)\,(z-z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z-z_0)^2}{2} + \ldots,$$

one can identify

$$\mathcal{J}(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle},$$

$$\mathrm{tr}\,\mathcal{J}^2(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}\,\frac{4N\Delta}{c}.$$

Tau function (continued)

But $\partial_{a_\mu} \ln \tau = \frac{1}{2} \mathrm{res}_{z=a_\mu} \mathrm{tr}\, \mathcal{J}^2(z)$ and

$$\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) T(z) \rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \rangle} = \sum_{\nu=1}^{n} \left\{ \frac{\tilde{\Delta}_\nu}{(z-a_\nu)^2} + \frac{1}{z-a_\nu} \partial_{a_\nu} \ln \left\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \right\rangle \right\}$$

which implies

$$\boxed{\tau(a) = \left\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \right\rangle^{\frac{2N\Delta}{c}}}$$

- in particular, for $c = 2N\Delta$ we have $\tilde{\Delta}_\nu = \frac{1}{2} \mathrm{tr}\, \mathcal{A}_\nu^2$

Example [SMJ, '79; Moore, '90] (N free complex fermions $\{\bar{\psi}_j\}$, $\{\psi_k\}$)

- $c = 2 \cdot N \cdot \frac{1}{2}$, current $J_{jk} = (\bar{\psi}_j \psi_k)$, $T = \frac{1}{2} \sum_k \left[(\bar{\psi}_k \, \partial \psi_k) - (\partial \bar{\psi}_k \, \psi_k) \right]$
- monodromy fields obtained by bosonization

$$\bar{\psi}_k = : e^{-i\phi_k} :, \qquad \psi_k = : e^{i\phi_k} :,$$

$$J_{jk} = \begin{cases} : e^{i(\phi_k - \phi_j)} :, & j \neq k, \\ i\, \partial \phi_k, & j = k, \end{cases} \qquad T = -\frac{1}{2} \sum_k \left(\partial \phi_k \, \partial \phi_k \right),$$

$$\mathcal{O}_{\mathcal{L}_\nu} = : e^{i \sum_k \lambda_{\nu,k} \phi_k^{(\nu)}} : .$$

- need N distinct bosonization schemes

Back to isomonodromy problem

Decompose $\mathcal{A}_\nu$'s as $\mathcal{A}_\nu = \frac{\operatorname{tr}\mathcal{A}_\nu}{N}\,\mathbf{1}_N + \hat{\mathcal{A}}_\nu$, then

$$\Phi_{\mathcal{A}}(z) = \prod_\nu \left(\frac{z-a_\nu}{z_0-a_\nu}\right)^{\frac{\operatorname{tr}\mathcal{A}_\nu}{N}} \Phi_{\hat{\mathcal{A}}}(z),$$

$$\mathcal{J}_{\mathcal{A}}(z) = \frac{1}{N}\sum_\nu \frac{\operatorname{tr}\mathcal{A}_\nu}{z-a_\nu}\,\mathbf{1}_N + \mathcal{J}_{\hat{\mathcal{A}}}(z),$$

$$\tau_{\mathcal{A}}(a) = \prod_{\mu<\nu}(a_\mu - a_\nu)^{\frac{\operatorname{tr}\mathcal{A}_\mu \operatorname{tr}\mathcal{A}_\nu}{N}}\,\tau_{\hat{\mathcal{A}}}(a).$$

Example (continued)

$$N \text{ complex fermions} = \hat{u}(1) \oplus \widehat{su}(N)_1$$

Fermion and monodromy fields factorize

$$\bar{\psi}_k = :e^{-i\phi_0/\sqrt{N}}: \otimes\, \hat{\bar{\varphi}}_k, \qquad \psi_k = :e^{i\phi_0/\sqrt{N}}: \otimes\, \hat{\varphi}_k,$$

$$\mathcal{O}_{\mathcal{L}_\nu} = :e^{\frac{i\operatorname{tr}\mathcal{A}_\nu}{\sqrt{N}}\phi_0}: \otimes\, \mathcal{O}_{\hat{\mathcal{L}}_\nu}$$

- dimension $\Delta = \frac{N-1}{2N}$ of $\{\hat{\bar{\varphi}}_k\}$ and $\{\hat{\varphi}_k\}$ agrees with $c_{\widehat{su}(N)_1} = N-1$
- dimension of $\mathcal{O}_{\hat{\mathcal{L}}_\nu}$ is equal to $\frac{1}{2}\operatorname{tr}\hat{\mathcal{A}}_\nu^2$
- tracelessness of $\mathcal{A}(z)$ corresponds to factoring out the $\hat{u}(1)$ piece

Conclusion: Isomonodromic tau function can be interpreted as a correlation function of primaries with dimensions $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2$ in a CFT with $c = N-1$.

Remark. For $N=2$ the dimension $\Delta = \frac{1}{4}$ of φ's and $\bar{\varphi}$'s corresponds to states degenerate at level 2, and the dimension 1 of $\{J_{jk}\}$ is degenerate at level 3. Hence the correlators

$$\mathcal{P}_{jk} = \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle,$$
$$\mathcal{Q}_{jk} = \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J_{jk}(z)\rangle,$$

have to satisfy linear PDEs of order 2 and 3, fixed by Virasoro symmetry.

Proposition. Under assumption $\operatorname{tr}\mathcal{A}(z)=0$, the matrices

$$\mathcal{P} = (z-z_0)^{-\frac{1}{2}}\tau\Phi, \qquad \mathcal{Q} = \tau\Phi^{-1}\partial_z\Phi,$$

satisfy

$$\partial_{zz}\mathcal{P} = \left\{\frac{1}{z-z_0}\,\partial_{z_0} + \frac{1}{4\,(z-z_0)^2} + \sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu} + \frac{\Delta_\nu}{(z-a_\nu)^2}\right)\right\}\mathcal{P},$$

$$\partial_{zzz}\mathcal{Q} = \left\{4\sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu z} + \frac{\Delta_\nu}{(z-a_\nu)^2}\,\partial_z\right) + 2\sum_\nu\left(\frac{1}{(z-a_\nu)^2}\,\partial_{a_\nu} + \frac{2\Delta_\nu}{(z-a_\nu)^3}\right)\right\}\mathcal{Q}.$$

Painlevé VI

PVI tau function is a 4-point correlator of monodromy fields,

$$\tau(t) = \langle \mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_1}(1)\mathcal{O}_{\mathcal{L}_\infty}(\infty)\rangle,$$

and these fields are Virasoro primaries with dimensions $\Delta_\nu = \theta_\nu^2$ in a $c = 1$ CFT.

- "conservation of monodromy" $\Rightarrow \{\varphi_k\}$ should have monodromy $\mathcal{M}_t\mathcal{M}_0$ around all fields in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$
- if $\mathcal{M}_t\mathcal{M}_0 = \mathcal{C}_{0t}^{-1}\begin{pmatrix} e^{2\pi i\sigma_{0t}} & 0 \\ 0 & e^{-2\pi i\sigma_{0t}}\end{pmatrix}\mathcal{C}_{0t}$, then it is natural to expect that the set of primaries in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$ consists of an infinite number of monodromy fields $\mathcal{O}_{\mathcal{L}_{0t}^{(n)}}$ with $n \in \mathbb{Z}$ and

$$\mathcal{L}_{0t}^{(n)} = \mathcal{C}_{0t}^{-1}\begin{pmatrix} \sigma_{0t}+n & 0 \\ 0 & -\sigma_{0t}-n\end{pmatrix}\mathcal{C}_{0t}$$

- inserting the OPE $\mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)$ into the correlator gives

$$\boxed{\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{\Delta_{\sigma+n}-\Delta_0-\Delta_t}\mathcal{B}(\boldsymbol{\theta}, \sigma+n, t)}$$

Computation of conformal blocks

1. direct (inversion of the Shapovalov form)
2. recursion relation [Zamolodchikov, '84]
3. combinatorial representations, conjectured in [Alday, Gaiotto, Tachikawa, '09] and proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]

Structure constants

Jimbo's asymptotic formula can be interpreted as a recurrence relation

$$\frac{C_{n\pm1}}{C_n}=\frac{\Gamma^2\left(1\mp 2(\sigma_{0t}+n)\right)}{\Gamma^2\left(1\pm 2(\sigma_{0t}+n)\right)}\prod_{\epsilon=\pm}\frac{\Gamma\left(1+\epsilon\theta_0+\theta_t\pm(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\pm(\sigma_{0t}+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t\mp(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\mp(\sigma_{0t}+n)\right)}\times$$
$$\times\frac{\left(\theta_0^2-(\theta_t\mp(\sigma_{0t}+n))^2\right)\left(\theta_\infty^2-(\theta_1\mp(\sigma_{0t}+n))^2\right)}{4\left(\sigma_{0t}+n\right)^2\left(1\pm 2(\sigma_{0t}+n)\right)^2}\left(-s_{0t}\right)^{\pm1}$$

with the solution in terms of Barnes functions

$$\boxed{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)=s_{0t}^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right)G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}}$$

Main claim

Complete expansion of Painlevé VI tau function near $t=0$ can be written as

$$\tau(t) = \text{const} \cdot \sum_{n\in\mathbb{Z}} C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta},\sigma_{0t}+n;t).$$

The function $\mathcal{B}(\boldsymbol{\theta},\sigma;t)$ is a power series in t which coincides with the general $c=1$ conformal block and is explicitly given by

$$\mathcal{B}\left(\boldsymbol{\theta},\sigma;t\right) = (1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right)t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\times$$

$$\times\prod_{(i,j)\in\mu}\frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}.$$

The structure constants $\{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)\}_{n\in\mathbb{Z}}$ can be written in terms of Barnes G-function,

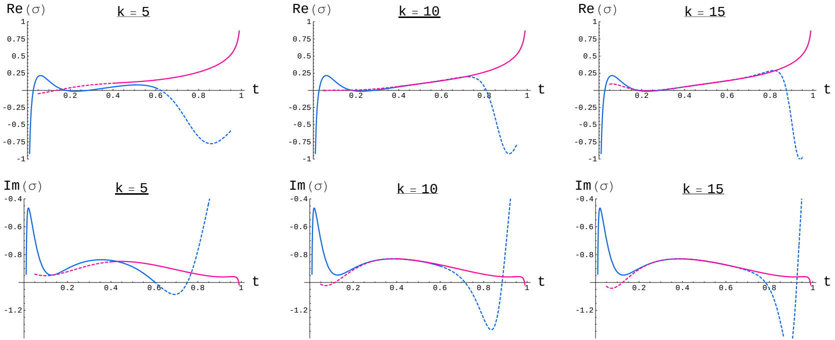
$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) = s_{0t}^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right)G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}$$

Remarks

- checked about 30 first terms in the asymptotic expansion of τ (up to level 10, ~ 500 bipartitions) in full generality
- to prove rigorously, it is sufficient to demonstrate two bilinear relations satisfied by $c=1$ conformal blocks
- expansions at $1, \infty$ are obtained by parameter change; for example, near $t=1$

$$\theta_0 \leftrightarrow \theta_1, \quad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p'_{01} = \omega_{01} - p_{01} - p_{0t}p_{1t}.$$

- series representations suitable for numerical evaluation of PVI functions



$$\begin{pmatrix} \theta_0 \\ \theta_t \\ \theta_1 \\ \theta_\infty \end{pmatrix} = \begin{pmatrix} 0.501790 + 0.216884i \\ 0.382251 + 0.723641i \\ 0.152700 + 0.358959i \\ 0.158518 + 0.674992i \end{pmatrix}, \quad \begin{pmatrix} \sigma_{0t} \\ \sigma_{1t} \end{pmatrix} = \begin{pmatrix} 0.837497 + 0.943080i \\ 0.411398 + 0.480375i \end{pmatrix}$$

Picard solutions

- $\omega_{0t} = \omega_{1t} = \omega_{01} = \omega_4 = 0$
- parameters can be Backlund transformed to $\boldsymbol{\theta}_{\text{Picard}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- dimensions $\Delta_\nu = \theta_\nu^2$ correspond to Ashkin-Teller conformal block [Zamolodchikov, '86]

$$\mathcal{B}\left(\boldsymbol{\theta}_{\text{Picard}}, \sigma; t\right) = \frac{\left(16t^{-1}q\right)^{\sigma^2}}{(1-t)^{\frac{1}{8}}\vartheta_3(0|\tau)}$$

where $q = e^{i\pi\tau}$, $\tau = \dfrac{iK'(t)}{K(t)}$ and

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}, \qquad K'(t) = K(1-t).$$

- structure constants C_n and parameter s_{0t} simplify to

$$C_n \sim 2^{-4(\sigma_{0t}+n)^2}(-s_{0t})^n, \qquad s = -e^{\pm 2\pi i \sigma_{1t}}$$

- conformal expansion $\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n, t)$ then gives theta function series so that finally

$$\tau_{\text{Picard}}(t) = \text{const}\cdot \frac{q^{\sigma_{0t}^2}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\,\frac{\vartheta_3\left(\sigma_{0t}\pi\tau \pm \sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

(this indeed coincides with Picard tau function [Kitaev, Korotkin, '98])

Coalescence diagram revisited

$$
\begin{array}{ccccccccc}
\mathrm{VI} & \longrightarrow & \mathrm{V} & \longrightarrow & \mathrm{III}_1 & \longrightarrow & \mathrm{III}_2 & \longrightarrow & \mathrm{III}_3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{IV} & \longrightarrow & \mathrm{II} & \longrightarrow & \mathrm{I} & &
\end{array}
$$

- can easily write similar expansions for Painlevé V and $\mathrm{III}_{1,2,3}$
- coalescence corresponds to decoupling of matter hypermultiplets

$$
\begin{array}{c} N_f = 4 \\ (P_{\mathrm{VI}}) \end{array} \xrightarrow{\mu_4 \to \infty} \begin{array}{c} N_f = 3 \\ (P_{\mathrm{V}}) \end{array} \xrightarrow{\mu_3 \to \infty} \begin{array}{c} N_f = 2 \\ (P_{\mathrm{III}_1}) \end{array} \xrightarrow{\mu_2 \to \infty} \begin{array}{c} N_f = 1 \\ (P_{\mathrm{III}_2}) \end{array} \xrightarrow{\mu_1 \to \infty} \begin{array}{c} \text{pure gauge theory} \\ (P_{\mathrm{III}_3}) \end{array}
$$

Application to SG_{ff} correlator

AGT series:

$$D(mr) = \sum_{n\in\mathbb{Z}} C_{\mathrm{SG}}(\nu+n,\nu'+n)\left(\frac{m^2r^2}{4}\right)^{(\nu+n)(\nu'+n)} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{\mathrm{SG}}_{\lambda,\mu}(\nu+n,\nu'+n)\left(\frac{m^2r^2}{4}\right)^{|\lambda|+|\mu|}$$

with

$$C_{\mathrm{SG}}(\nu,\nu') = G\left[\begin{array}{c} 1+\nu, 1-\nu, 1+\nu', 1-\nu' \\ 1+\nu+\nu', 1-\nu-\nu' \end{array}\right]$$

$$\mathcal{B}^{\mathrm{SG}}_{\lambda,\mu}(\nu,\nu') = \prod_{(i,j)\in\lambda} \frac{(i-j+\nu)(i-j+\nu')}{h^2_\lambda(i,j)\left(\lambda'_j - i + \mu_i - j + 1 + \nu + \nu'\right)^2} \prod_{(i,j)\in\mu} \frac{(i-j-\nu)(i-j-\nu')}{h^2_\mu(i,j)\left(\mu'_j - i + \lambda_i - j + 1 - \nu - \nu'\right)^2}$$

- Holomorphic conformal blocks describe CPT expansion of massive theory!

Conclusions

1. Painlevé VI, V, III tau functions are Fourier transforms of $c = 1$ conformal blocks and their irregular analogs
2. AGT combinatorial formulas provide series representations for general solutions of Painlevé VI, V, III and an efficient tool of numerical computation

More questions

1. connection problem for Painlevé tau functions/fusion matrix of $c = 1$ conformal blocks ✓
2. increase rank/genus/number of singular points
3. rigorous proof?