Onsager's approach and correlation functions of the XXZ open spin chain

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Dijon - September 2013

Part I: arXiv:1211.6304, Nucl.Phys.B (2013)
+ in progress with S. Belliard (Montpellier)
Part II: to appear with T. Kojima (Yamagata)

Lattice quantum integrable models : aims and approaches

Finding **analytical expressions** for observables or related objects (energy spectrum, eigenstates states, scatterring amplitudes, correlation functions, form factors) is one of the main objective, giving an access to **non-perturbative** predictions of the phenomena.

Most of lattice quantum integrable models have been studied using the **following approaches** based on :

- Bethe ansatz : coordinate/functional/algebraic
- Sklyanin's separation of variables (SOV)
- Hidden symmetries and $q-$vertex operators (VOs)
- Fermionic basis
- Onsager algebra and $q-$extension $\rightarrow$ **HERE**

$\triangleright$ **XXZ open spin chain : interesting example**

$\rightarrow$ **Some approaches : problems for general boundary conds.**
$\rightarrow$ **Onsager's approach : alternative to ABA/SOV**
$\rightarrow$ **'Experimental frame' following 'CFT strategy'**

Motivations I : solving the open XXZ chain (old/recent/new results)

$$H = \sum_{k=1}^{\infty} \left(\sigma_x^k \sigma_x^{k+1} + \sigma_y^k \sigma_y^{k+1} + \Delta \sigma_z^k \sigma_z^{k+1} \right) + \beta \sigma_z^1 + \alpha \left(k_+ \sigma_+^1 + k_- \sigma_-^1 \right) .$$

- *Boundary parameters :* $\alpha, \beta \in \mathbb{R}$, $k_\pm \in \mathbb{C}$;
- *Anisotropy parameter :* $\Delta = (q + q^{-1})/2$.

XXZ open chain	**Hidd. sym.**	$E/\|states>$	**Correls.**	**Form F.**	**Approach**
Diagonal bcs.		OK	OK	OK	q-VO$^{(*)}$ (1994)
$k_\pm = 0$		OK	OK	OK	ABA$^{(**)}$ (2010)
	OK	**OK**	**OK**	**OK**	q-Onsager/VO
Triangular bcs.		possible$^{(***)}$	?	?	ABA
$k_+ = 0$ or $k_- = 0$	**OK**	**OK**	**OK**	**OK**	q-Onsager/VO
Generic bcs	**OK**	in prog.	possible	possible	q-Onsager/VO

Refs : $^{(*)}$ *Jimbo-Kedem-Kojima-Konno-Miwa (1994)*
$^{(**)}$ *Sklyanin (1988)* $^{(***)}$ XXX case : *Belliard-Crampé-Ragoucy (2013)*
$^{(**)}$ *Kitanine-Kozlowski-Maillet-Niccoli-Slavnov-Terras (2007)*

REMARK on alternative approaches : recent works using functional relations (Galleas, 2007) or Sklyanin's SOV approach (Niccoli et al., 2012)

Motivations II : open XXZ chain as an experimental ground

Conformal field theory : For the class of quantum integrable models in the continuum with infinite dimensional symmetry (local). *Ex : Minimal models*

- Virasoro algebra : $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$,
- Stress energy tensor : $T(u) \to L_n$,
- Hamiltonian $\to$ spectral problem for $L_0 + \overline{L_0}$,
- Basis : $L_{-n_1}...L_{-n_N}|\Omega\rangle$,
- Correlation functions $\leftrightarrow$ hypergeometric functions

Towards a similar program for a class of non-conformal integrable models ?

Onsager's framework : For the class of quantum integrable models in the continuum or lattice with a (non-local) infinite dimensional $q-$Onsager spectrum generating algebra. *Ex : Ising, XY, superintegrable chiral Potts, XXZ open chain,...*

- $q-$Onsager algebra : generators $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$ $k \geq 0$
- Transfer matrix : $t(u) \to \mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$,
- Hamiltonian $\to$ spectral problem for $\kappa\mathcal{W}_0 + \kappa^*\mathcal{W}_1 + \bar{k}_+\mathcal{G}_1 + \bar{k}_-\tilde{\mathcal{G}}_1$,
- Basis : $\mathcal{W}_{-k_1}...\mathcal{W}_{-k_N}\mathcal{G}_{p_1+1}...\mathcal{G}_{p_P+1}\mathcal{W}_{l_M+1}...\mathcal{W}_{l_1+1}|\Omega\rangle$,
- Correlation functions $\leftrightarrow$ generalization of Askey-Wilson's $q-$hypergeometric fcts.

Step 1 : The finite open chain in Onsager's framework

For **general integrable** boundary conditions and q, its Hamiltonian is given by :

$$H^{(N)}_{XXZ} = \sum_{k=1}^{N-1}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) + \frac{(q-q^{-1})}{2}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_3^1 + \frac{2}{(\epsilon_++\epsilon_-)}\left(k_+\sigma_+^1 + k_-\sigma_-^1\right) + \frac{(q-q^{-1})}{2}\frac{(\bar\epsilon_+-\bar\epsilon_-)}{(\bar\epsilon_++\bar\epsilon_-)}\sigma_3^N + \frac{2}{(\bar\epsilon_++\bar\epsilon_-)}\left(\bar k_+\sigma_+^N + \bar k_-\sigma_-^N\right), \quad \sigma_\pm = (\sigma_1 \pm i\sigma_2)/2.$$

- $\epsilon_\pm, k_\pm$ (resp. $\bar\epsilon_\pm, \bar k_\pm$) denote that the right (resp. left) boundary parameters

The corresponding **transfer matrix** can be formulated in :

▷ **Sklyanin's framework** (Sklyanin, 1988) ⇒ **Standard approach** (ABA,SOV,qVO)

$$t^{(N)}(\zeta) = tr_0\Big[\underbrace{K_+(\zeta)}_{K-matrix}\bar R_{0N}(\zeta)\cdots\bar R_{01}(\zeta)K_-(\zeta)\underbrace{\bar R_{01}(\zeta)}_{R-matrix}\cdots\bar R_{0N}(\zeta)\Big]$$

▷ **Onsager's framework** (P.B-Koizumi, 2007) ⇒ **Approach here considered**

$$t^{(N)}(\zeta) = \sum_{k=0}^{N-1}\mathcal{F}_{2k+1}(\zeta)\underbrace{\mathcal{I}^{(N)}_{2k+1}}_{q-Dolan-Grady\ hierarchy} + \mathcal{F}_0(\zeta)\, I\!I^{(N)}$$

▷ Onsager's framework

The **transfer matrix** in Onsager's presentation is :

$$t^{(N)}(\zeta) = \sum_{k=0}^{N-1} \underbrace{\mathcal{F}_{2k+1}(\zeta)}_{rational fcts.} \; \mathcal{I}_{2k+1}^{(N)} + \mathcal{F}_0(\zeta)\; I\!I^{(N)} \quad \text{with} \quad [\mathcal{I}_{2k+1}^{(N)}, \mathcal{I}_{2l+1}^{(N)}] = 0$$

act on $\mathcal{V}^{(\mathcal{N})} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2}_{N times}$

Result 1. The $q-$**deformed Dolan-Grady hierarchy** :

→ For **non-diagonal bcs at site** N. (P.B-Koizumi, 2007)

$$\mathcal{I}_{2k+1}^{(N)} = \epsilon_+ \mathcal{W}_{-k}^{(N)} + \epsilon_- \mathcal{W}_{k+1}^{(N)} + \frac{1}{q^2-q^{-2}}\left(\frac{k_-}{\bar{k}_-}\mathcal{G}_{k+1}^{(N)} + \frac{k_+}{\bar{k}_+}\tilde{\mathcal{G}}_{k+1}^{(N)}\right)$$

→ For **diagonal bcs at site** N.

$$\mathcal{I}_{2k+1}^{(N)} = \epsilon_+ \mathcal{K}_{-k}^{(N)} + \epsilon_- \mathcal{K}_{k+1}^{(N)} + \frac{1}{q^2-q^{-2}}\left(k_- \mathcal{Z}_{k+1}^{(N)} + k_+ \tilde{\mathcal{Z}}_{k+1}^{(N)}\right)$$

Result 2. The **spectrum generating algebra :** (finite case : + Davies' relations)

→ For **non-diag. bcs. (site** N**)** : $\mathsf{W}_0, \mathsf{W}_1$: $q-$**Onsager algebra** $\sim \mathcal{A}_q$
$\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1}$ $k \geq 1$ 'descendents'

→ For **diag. bcs. at (site** N**)** : $\mathsf{K}_0, \mathsf{K}_1, \mathcal{Z}_1, \tilde{\mathcal{Z}}_1$: **augmented** $q-$**Onsager alg.** $\sim \mathcal{A}_q^{diag}$
$\mathsf{K}_{-k}, \mathsf{K}_{k+1}, \mathsf{Z}_{k+1}, \tilde{\mathsf{Z}}_{k+1}$ $k \geq 1$ 'descendents'

Step 2 : The thermodynamic limit and current algebra

The **Hamiltonian of the half-infinite XXZ** spin chain with an integrable boundary can be considered as the thermodynamic limit $N \to \infty$ of the finite XXZ open spin chain :

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\Big(\sigma_1^{k+1}\sigma_1^k+\sigma_2^{k+1}\sigma_2^k+\Delta\sigma_3^{k+1}\sigma_3^k\Big)-\frac{q-q^{-1}}{4}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_3^1-\frac{1}{(\epsilon_++\epsilon_-)}(k_+\sigma_+^1+k_-\sigma_-^1)$$

What is the analog of Onsager's formulation in the thermodynamic limit ?

- **Transfer matrix :** $t^{(N)}(\zeta) \longrightarrow t^{(\mathcal{V})}(\zeta)$
- $q-$**Dolan-Grady hierarchy :** $\mathcal{I}^{(N)}_{2k+1} \longrightarrow \mathcal{I}^{(\mathcal{V})}(\zeta)$

$$\mathcal{I}(-\zeta^{-1}q^{-1}) = \epsilon_+\mathcal{W}_-(\zeta)+\epsilon_-\mathcal{W}_+(\zeta)+\tfrac{1}{q^2-q^{-2}}(k_-\mathcal{Z}_-(\zeta)+k_+\mathcal{Z}_+(\zeta)) \ .$$

- **Spectrum generating algebra :** Davies' relations $\sum_k \alpha_k\mathcal{W}_k = 0$ disappear.

Truncated sums of the finite case $\longrightarrow O_q(\widehat{sl_2})$ currents $\{\mathcal{W}_\pm(\zeta),\mathcal{Z}_\pm(\zeta)\}$ of the form

$$\Upsilon(\zeta) = \sum_{k\in\mathbb{Z}_+}\Upsilon_k U(\zeta)^{-k-1} \ , \qquad U(\zeta) = (q\zeta^2+q^{-1}\zeta^{-2})/(q+q^{-1}) \ .$$

$O_q(\widehat{sl_2})$ current algebra : The **transfer matrix** of the XXZ open spin chain with **general boundary conditions** can be expressed in terms of $O_q(\widehat{sl_2})$ currents with relations :

$$
\begin{aligned}
&\big[\mathcal{W}_\pm(\zeta),\mathcal{W}_\pm(\xi)\big]=0\ ,\quad \big[\mathcal{W}_+(\zeta),\mathcal{W}_-(\xi)\big]+\big[\mathcal{W}_-(\zeta),\mathcal{W}_+(\xi)\big]=0\ ,\\
&(U(\zeta)-U(\xi))\big[\mathcal{W}_\pm(\zeta),\mathcal{W}_\mp(\xi)\big]=\frac{(q-q^{-1})}{(q+q^{-1})^3}\left(\mathcal{Z}_\pm(\zeta)\mathcal{Z}_\mp(\xi)-\mathcal{Z}_\pm(\xi)\mathcal{Z}_\mp(\zeta)\right)\ ,\\
&\mathcal{W}_\pm(\zeta)\mathcal{W}_\pm(\xi)-\mathcal{W}_\mp(\zeta)\mathcal{W}_\mp(\xi)+\frac{1}{(q^2-q^{-2})(q+q^{-1})^2}\big[\mathcal{Z}_\pm(\zeta),\mathcal{Z}_\mp(\xi)\big]\\
&\qquad\qquad+\frac{1-U(\zeta)U(\xi)}{U(\zeta)-U(\xi)}\left(\mathcal{W}_\pm(\zeta)\mathcal{W}_\mp(\xi)-\mathcal{W}_\pm(\xi)\mathcal{W}_\mp(\zeta)\right)=0\ ,\\
&U(\zeta)\big[\mathcal{Z}_\mp(\xi),\mathcal{W}_\pm(\zeta)\big]_q-U(\xi)\big[\mathcal{Z}_\mp(\zeta),\mathcal{W}_\pm(\xi)\big]_q\\
&\qquad\qquad-(q-q^{-1})\left(\mathcal{W}_\mp(\zeta)\mathcal{Z}_\mp(\xi)-\mathcal{W}_\mp(\xi)\mathcal{Z}_\mp(\zeta)\right)=0,\\
&U(\zeta)\big[\mathcal{W}_\mp(\zeta),\mathcal{Z}_\mp(\xi)\big]_q-U(\xi)\big[\mathcal{W}_\mp(\xi),\mathcal{Z}_\mp(\zeta)\big]_q\\
&\qquad\qquad-(q-q^{-1})\left(\mathcal{W}_\pm(\zeta)\mathcal{Z}_\mp(\xi)-\mathcal{W}_\pm(\xi)\mathcal{Z}_\mp(\zeta)\right)=0\ ,\\
&\big[\mathcal{Z}_\epsilon(\zeta),\mathcal{W}_\pm(\xi)\big]+\big[\mathcal{W}_\pm(\zeta),\mathcal{Z}_\epsilon(\xi)\big]=0\ ,\quad \forall\epsilon=\pm\\
&\big[\mathcal{Z}_\pm(\zeta),\mathcal{Z}_\pm(\xi)\big]=0\ ,\qquad \big[\mathcal{Z}_+(\zeta),\mathcal{Z}_-(\xi)\big]+\big[\mathcal{Z}_-(\zeta),\mathcal{Z}_+(\xi)\big]=0\ .
\end{aligned}
$$

In **Onsager's formulation**, the **transfer matrix** reads : $\quad t^{(\mathcal{V})}(\zeta)=g\frac{(\zeta^2-\zeta^{-2})}{\rho(\zeta)}\mathcal{I}^{(\mathcal{V})}(\zeta)$

with $\quad \mathcal{I}(-\zeta^{-1}q^{-1})=\epsilon_+\mathcal{W}_-(\zeta)+\epsilon_-\mathcal{W}_+(\zeta)+\frac{1}{q^2-q^{-2}}\left(k_-\mathcal{Z}_-(\zeta)+k_+\mathcal{Z}_+(\zeta)\right)$

$\Longrightarrow$ **Currents' bosonization ? Connection with $U_q(\widehat{sl_2})$ $q-$vertex operators ?**

Step 3 : Central extension and bosonization

$O_q(\widehat{sl_2})$ **currents** admit formal expansions in the **symmetric** spectral parameter $U(\zeta)$ according to the choice of boundary conditions at $N=\infty$. Two homomorphisms :

$\rightarrow$ For **non-diagonal bcs at** $N=\infty$**.**

$$\begin{pmatrix}\mathcal{W}_+(\zeta)\\ \mathcal{W}_-(\zeta)\end{pmatrix} \rightarrow \sum_{k\in\mathbb{Z}_+}\begin{pmatrix}\mathsf{W}_{-k}\\ \mathsf{W}_{k+1}\end{pmatrix}U(\zeta)^{-k-1}\ , \qquad U(\zeta)=(q\zeta^2+q^{-1}\zeta^{-2})/(q+q^{-1})$$

$$\begin{pmatrix}\mathcal{Z}_+(\zeta)\\ \mathcal{Z}_-(\zeta)\end{pmatrix} \rightarrow \frac{1}{\bar{k}_\mp}\sum_{k\in\mathbb{Z}_+}\begin{pmatrix}\mathsf{G}_{k+1}\\ \tilde{\mathsf{G}}_{k+1}\end{pmatrix}U(\zeta)^{-k-1}+\begin{pmatrix}\bar{k}_+\\ \bar{k}_-\end{pmatrix}\frac{(q+q^{-1})^2}{(q-q^{-1})}\ .$$

• The first modes satisfy the $q-$**Onsager algebra** with $\rho=(q+q^{-1})^2\bar{k}_+\bar{k}_-$.

$$[\mathsf{W}_0,[\mathsf{W}_0,[\mathsf{W}_0,\mathsf{W}_1]_q]_{q^{-1}}]=\rho[\mathsf{W}_0,\mathsf{W}_1]\ ,[\mathsf{W}_1,[\mathsf{W}_1,[\mathsf{W}_1,\mathsf{W}_0]_q]_{q^{-1}}]=\rho[\mathsf{W}_1,\mathsf{W}_0]$$

• **Realization** in terms of $U_q(\widehat{sl_2})$ Chevalley elements (P.B, 2004) :

$$\mathsf{W}_0=\bar{k}_+e_1+\bar{k}_-q^{-1}f_1q^{h_1}+\bar{\epsilon}_+q^{h_1}\ ,$$
$$\mathsf{W}_1=\bar{k}_-e_0+\bar{k}_+q^{-1}f_0q^{h_0}+\bar{\epsilon}_-q^{h_0}\ .$$

$\Rightarrow$ **Level 1 infinite dim. rep (**$q-$**vertex operators)**

→ For **diagonal bcs at** $N = \infty$.

$$\begin{pmatrix}\mathcal{W}_+(\zeta)\\ \mathcal{W}_-(\zeta)\end{pmatrix} \rightarrow \sum_{k\in\mathbb{Z}_+}\begin{pmatrix}\mathsf{K}_{-k}\\ \mathsf{K}_{k+1}\end{pmatrix} U(\zeta)^{-k-1}\ , \qquad U(\zeta) = (q\zeta^2 + q^{-1}\zeta^{-2})/(q+q^{-1})$$

$$\begin{pmatrix}\mathcal{Z}_+(\zeta)\\ \mathcal{Z}_-(\zeta)\end{pmatrix} \rightarrow \sum_{k\in\mathbb{Z}_+}\begin{pmatrix}\mathsf{Z}_{k+1}\\ \tilde{\mathsf{Z}}_{k+1}\end{pmatrix} U(\zeta)^{-k-1}\ .$$

• The first modes satisfy the **augmented $q-$Onsager algebra** (Ito-Terwilliger, 2010) :

$$[\mathsf{K}_0,\mathsf{K}_1] = 0\ , \qquad \rho_{diag} = \frac{(q^3-q^{-3})(q^2-q^{-2})^3}{q-q^{-1}}$$

$$\mathsf{K}_0\mathsf{Z}_1 = q^{-2}\mathsf{Z}_1\mathsf{K}_0\ ,\ \mathsf{K}_0\tilde{\mathsf{Z}}_1 = q^{2}\tilde{\mathsf{Z}}_1\mathsf{K}_0\ ,\ \mathsf{K}_1\mathsf{Z}_1 = q^{2}\mathsf{Z}_1\mathsf{K}_1\ ,\ \mathsf{K}_1\tilde{\mathsf{Z}}_1 = q^{-2}\tilde{\mathsf{Z}}_1\mathsf{K}_1\ ,$$

$$\big[\mathsf{Z}_1,\big[\mathsf{Z}_1,\big[\mathsf{Z}_1,\tilde{\mathsf{Z}}_1\big]_q\big]_{q^{-1}}\big] = \rho_{diag}\mathsf{Z}_1(\mathsf{K}_1\mathsf{K}_1 - \mathsf{K}_0\mathsf{K}_0)\mathsf{Z}_1,$$

$$\big[\tilde{\mathsf{Z}}_1,\big[\tilde{\mathsf{Z}}_1,\big[\tilde{\mathsf{Z}}_1,\mathsf{Z}_1\big]_q\big]_{q^{-1}}\big] = \rho_{diag}\tilde{\mathsf{Z}}_1(\mathsf{K}_0\mathsf{K}_0 - \mathsf{K}_1\mathsf{K}_1)\tilde{\mathsf{Z}}_1\ .$$

• **Realization** in terms of $U_q(\widehat{sl_2})$ Chevalley elements :

$$\mathsf{K}_0 = \bar{\epsilon}_+ q^{h_1}\ , \qquad \mathsf{K}_1 = \bar{\epsilon}_- q^{h_0}\ ,$$
$$\mathsf{Z}_1 = (q^2-q^{-2})\big(\bar{\epsilon}_+ q^{-1} e_0 q^{h_1} + \bar{\epsilon}_- f_1 q^{h_1+h_0}\big)\ ,$$
$$\tilde{\mathsf{Z}}_1 = (q^2-q^{-2})\big(\bar{\epsilon}_- q^{-1} e_1 q^{h_0} + \bar{\epsilon}_+ f_0 q^{h_1+h_0}\big)$$

⇒ Level 1 infinite dim. rep ($q-$vertex operators)

$O_q(\widehat{sl_2})$ $q-$**vertex operators** of type I and type II can be considered :

$$\chi(\zeta): \quad \mathcal{V} \to \mathcal{V} \otimes \underbrace{\mathcal{V}_\zeta}_{2\ dim.eval.rep.} \quad , \quad \overline{\chi}(\zeta): \quad \mathcal{V} \to \mathcal{V}_\zeta \otimes \mathcal{V} \ .$$

$$\begin{aligned} \text{Type } I: \quad & \chi(\zeta)\circ a &=& \ (id \times \pi_\zeta)\big[\delta(a)\big]\circ\chi(\zeta)\ , \\ \text{Type } II: \quad & \overline{\chi}(\zeta)\circ a &=& \ (\pi_\zeta \times id)\big[\delta(a)\big]\circ\overline{\chi}(\zeta) \qquad \forall a\in\mathcal{A}_q \text{ or } \mathcal{A}_q^{diag}\ . \end{aligned}$$

→ For **non-diagonal bcs at** ∞ **:** the coaction map δ of $\mathcal{A}_q$ ($q-$Onsager) is :

$$\begin{aligned} \delta(\mathsf{W}_0) &= (\bar{k}_+ e_1 + \bar{k}_- q^{-1} f_1 q^{h_1})\otimes 1 + q^{h_1}\otimes \mathsf{W}_0\ , \\ \delta(\mathsf{W}_1) &= (\bar{k}_- e_0 + \bar{k}_+ q^{-1} f_0 q^{h_0})\otimes 1 + q^{h_0}\otimes \mathsf{W}_1\ . \end{aligned}$$

→ For **diagonal bcs at** ∞ **:** the coaction map δ of $\mathcal{A}_q^{diag}$ (augmented $q-$Onsager) is :

$$\begin{aligned} &\delta(\mathsf{K}_0) = q^{h_1}\otimes \mathsf{K}_0\ , \qquad \delta(\mathsf{K}_1) = q^{h_0}\otimes \mathsf{K}_1\ , \\ &\delta(\mathsf{Z}_1) = q^{h_0+h_1}\otimes \mathsf{Z}_1 + (q^2-q^{-2})\big(q^{-1}e_0q^{h_1}\otimes \mathsf{K}_0 + f_1 q^{h_0+h_1}\otimes \mathsf{K}_1\big)\ , \\ &\delta(\tilde{\mathsf{Z}}_1) = q^{h_0+h_1}\otimes \tilde{\mathsf{Z}}_1 + (q^2-q^{-2})\big(f_0 q^{h_0+h_1}\otimes \mathsf{K}_0 + q^{-1}e_1q^{h_0}\otimes \mathsf{K}_1\big)\ . \end{aligned}$$

SOLUTION : Type I : $\chi(\zeta)\to\Phi^{(1-i,i)}(\zeta): \quad V(\Lambda_i)\to V(\Lambda_{1-i})\otimes\mathcal{V}_\zeta\ ,$

$(-1<q<0)$ Type II : $\overline{\chi}(\zeta)\to\Psi^{*(1-i,i)}(\zeta): \quad \underbrace{V(\Lambda_i)}_{U_q(\widehat{sl_2})\ level\ 1\ module} \to \mathcal{V}_\zeta\otimes V(\Lambda_{1-i})\ .$

Generalization to all modes of $O_q(\widehat{sl_2})$ currents :

The action of the currents on the $q-$vertex operators is obtained using the coaction maps. For instance ($\chi(\zeta) \to \Phi^{(1-i,i)}(\zeta) =$ Type I) :

$$\mathcal{W}_-(\zeta)\chi_-(v) \sim (q^{-1}U(\zeta) - U(v^{-1}\sqrt{q}))\chi_-(v)\mathcal{W}_-(\zeta) + q\frac{q-q^{-1}}{q+q^{-1}}\chi_-(v)\mathcal{W}_+(\zeta)$$
$$- v\, q^{-1}\frac{(q-q^{-1})}{(q+q^{-1})^2}\chi_+(v)\mathcal{Z}_-(\zeta)\ ,$$
$$\mathcal{W}_+(\zeta)\chi_-(v) \sim (q\,U(\zeta) - U(\sqrt{q}\,v^{-1}))\chi_-(v)\mathcal{W}_+(\zeta) - q^{-1}\frac{q-q^{-1}}{q+q^{-1}}\chi_-(v)\mathcal{W}_-(\zeta)$$
$$-v^{-1}q\,\frac{(q-q^{-1})}{(q+q^{-1})^2}\chi_+(v)\mathcal{Z}_-(\zeta)$$
$$\mathcal{Z}_-(\zeta)\chi_-(v) \sim \chi_-(v)\mathcal{Z}_-(\zeta)$$
$$\mathcal{Z}_+(\zeta)\chi_-(v) \sim (U(\zeta) - U(v\,q^{-1}))\chi_-(v)\mathcal{Z}_+(\zeta)$$
$$- (q^2 - q^{-2})(v\,q^{-1}\,U(\zeta) - v^{-1}q)\chi_+(v)\mathcal{W}_+(\zeta)$$
$$- (q^2 - q^{-2})(v^{-1}q\,U(\zeta) - v\,q^{-1})\chi_+(v)\mathcal{W}_-(\zeta)\ .$$

$\Longrightarrow$ **Realizations of $\mathcal{W}_\pm(\zeta), \mathcal{Z}_\pm(\zeta)$ in terms of $q-$vertex operators**

$$\mathcal{W}_\pm^{(i)}(\zeta) \to \frac{\zeta q \Phi_\mp^{(i,1-i)}(\zeta)\Phi_\pm^{(1-i,i)}(-\zeta^{-1}q^{-1}) + \zeta^{-1}q^{-1}\,\Phi_\pm^{(i,1-i)}(\zeta)\Phi_\mp^{(1-i,i)}(-\zeta^{-1}q^{-1})}{\zeta^2q^2 - \zeta^{-2}q^{-2}}\ ,$$
$$\mathcal{Z}_\pm^{(i)}(\zeta) \to (q+q^{-1})\Phi_\mp^{(i,1-i)}(\zeta)\Phi_\mp^{(1-i,i)}(-\zeta^{-1}q^{-1})\ .$$

REMARK : $\exists$ Alternative derivation (analog of Miki's formula, P.B-Belliard, 2011)

Step 4 : Currents' eigenvectors and a strategy for the spectral problem of H

- The **transfer matrix** : linear combination of the currents $\mathcal{W}_{\pm}^{(i)}(\zeta), \mathcal{Z}_{\pm}^{(i)}(\zeta)$
- **Realizations** of $\mathcal{W}_{\pm}^{(i)}(\zeta), \mathcal{Z}_{\pm}^{(i)}(\zeta)$ in terms of $q-$bosons known

Hamiltonian of the half-infinite XXZ spin chain with general integrable boundary cds. :

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) - \frac{(q-q^{-1})}{4}\frac{(\epsilon_+ - \epsilon_-)}{(\epsilon_+ + \epsilon_-)}\sigma_3^1 - \frac{(k_+\sigma_+^1 + k_-\sigma_-^1)}{(\epsilon_+ + \epsilon_-)}$$

Eigenstates (massive reg. $-1 < q < 0$) ? $\Rightarrow$ $q-$Onsager algebra rep. theor.

KEY POINT 1 : $\exists$ an homomorhism from the $q-$**Onsager algebra** with fundamental generators $\mathsf{W}_0, \mathsf{W}_1$ to the **augmented** $q-$**Onsager algebra** with fundamental generators $\mathsf{K}_0, \mathsf{K}_1, \mathcal{Z}_1, \tilde{\mathcal{Z}}_1$ (*Terwilliger-Ito, 2010*)

KEY POINT 2 : $\exists$ a basis of the aug. $q-$Onsager algebra (*Terwilliger-Ito, 2010*) : monomials in the fundamental generators the **augmented** $q-$**Onsager** $\Rightarrow$

Basis $\begin{pmatrix} \mathsf{K}_0^n \mathsf{K}_1^m \mathcal{Z}_1^{\lambda_0} \tilde{\mathcal{Z}}_1^{\lambda_1} \ldots \mathcal{Z}_1^{\lambda_{r(even)}} \\ \mathsf{K}_0^n \mathsf{K}_1^m \mathcal{Z}_1^{\lambda_0} \tilde{\mathcal{Z}}_1^{\lambda_1} \ldots \tilde{\mathcal{Z}}_1^{\lambda_{r(odd)}} \end{pmatrix} |\Omega\rangle$ iff $\lambda_0 < \lambda_1 < \cdots < \lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_r$

→ **Determination of the initial states $|\Omega\rangle \in |B_\pm\rangle$**

- **Fundamental eigenvectors :** $\mathcal{W}_\pm^{(i)}(\zeta)|B_\pm\rangle = \lambda(\zeta)|B_\pm\rangle$

$$|B_+\rangle = e^{\tilde{F}_0}|0\rangle \in V(\Lambda_0) \quad \text{and} \quad |B_-\rangle = e^{\alpha/2}e^{\tilde{F}_0}|0\rangle \in V(\Lambda_1)$$

where $\tilde{F}_0 = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{nq^{6n}}{[2n][n]} a_{-n}^2 - \sum_{n=1}^{\infty} \frac{q^{5n}(1-q^{2n})}{[4n]} a_{-2n}$.

Analog of tridiagonal pair's W_0, W_1 dual eigenbasis !

- **Spectrum :** $\lambda(-\zeta^{-1}q^{-1}) = \frac{1}{g}\frac{\zeta}{\zeta^2-\zeta^{-2}}\frac{\delta(\zeta^2)}{\delta(\zeta^{-2})}$, $\delta(z) = \frac{(q^6z^2;q^8)_\infty}{(q^8z^2;q^8)_\infty}$.

→ **Generalization :**

More generally, **two families of eigenstates** of $\mathcal{W}_\pm^{(i)}(\zeta)$ can be constructed using the properties of type II $q-$vertex operators and starting from $|B_\pm\rangle$:

$$\mathcal{W}_\pm^{(i)}(-\zeta^{-1}q^{-1})\underbrace{\Psi_{\mu_1}^*(\xi_1)\ldots\Psi_{\mu_m}^*(\xi_m)}_{TypeII\ VOs}|B_\pm\rangle = \underbrace{\lambda(-\zeta^{-1}q^{-1};\xi_1,\ldots,\xi_m)}_{spectrum}\Psi_{\mu_1}^*(\xi_1)\ldots\Psi_{\mu_m}^*(\xi_m)|B_\pm\rangle$$

with $\lambda(\zeta;\xi_1,\ldots,\xi_m) = \prod_{j=1}^m \tau(\zeta/\xi_j)\tau(\zeta\xi_j)\lambda(\zeta)$, $\tau(\zeta) = \zeta^{-1}\frac{\Theta_{q^4}(q\zeta^2)}{\Theta_{q^4}(q\zeta^{-2})}$

$$\Theta_p(z) = (z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty \ .$$

Application I : the diagonal case revisited

The **Hamiltonian** reads :

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) - \frac{q-q^{-1}}{4}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_3^1$$

$\longrightarrow$ Solution using $q-$VO by *Jimbo-Kedem-Kojima-Konno-Miwa, 1994*

In **Onsager's framework**, the **transfer matrix** is :

$$t^{(i)}(\zeta) = g\frac{(\zeta^2-\zeta^{-2})}{\rho(\zeta)}\left(\epsilon_+\mathcal{W}_-^{(i)}(\zeta) + \epsilon_-\mathcal{W}_+^{(i)}(\zeta)\right).$$

• **Fundamental eigenstates** of the transfer matrix are derived from the fundamental eigenstates of $O_q(\widehat{sl_2})$ currents, using level 1 infinite dimensional representations.

$$|0\rangle_B \equiv e^{f(v)}\underbrace{|B_+\rangle}_{\mathcal{W}_+(\zeta)\ eigenstate} \quad \text{and} \quad |1\rangle_B \equiv e^{-f(-q^{-1}v^{-1})}\underbrace{|B_-\rangle}_{\mathcal{W}_-(\zeta)\ eigenstate}$$

$$\text{with} \quad f(v) = -\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{7n/2}v^{2n} \quad \text{and} \quad v^2 = r = -\epsilon_+/\epsilon_-.$$

• **All eigenstates :** the solution of the spectral problem is (cf. Jimbo et al.) :

$$t^{(i)}(\zeta)\underbrace{\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)}_{TypeII\ q-VO}|i\rangle_B = \Lambda^{(i)}(\zeta;\xi_1,...,\xi_m;r)\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|i\rangle_B$$

Application I : the triangular case $k_- = 0$ (new)

The **Hamiltonian** reads :

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) - \frac{q-q^{-1}}{4}\frac{(\epsilon_+ - \epsilon_-)}{(\epsilon_+ + \epsilon_-)}\sigma_3^1 - \frac{k_+}{(\epsilon_+ + \epsilon_-)}\sigma_+^1$$

$\longrightarrow$ Up to now, no solution using $q-$VO was constructed ('symmetry' guide unknown for the construction of eigenstates)

In **Onsager's framework**, the **transfer matrix** is :

$$t^{(i)}(\zeta) = g\frac{(\zeta^2-\zeta^{-2})}{\rho(\zeta)}\left(\epsilon_+\mathcal{W}_-^{(i)}(\zeta) + \epsilon_-\mathcal{W}_+^{(i)}(\zeta) + \frac{k_+}{q^2-q^{-2}}\mathcal{Z}_+^{(i)}(\zeta)\right).$$

SOLUTION : Starting from $|\Omega\rangle \in \{|0\rangle_B, |1\rangle_B\}$ (diagonal solution), one is looking for a **combination of monomials** in $\mathcal{Z}_1, \tilde{\mathcal{Z}}_1$ acting on $|\Omega\rangle$.

$$|+;0\rangle_B = \underbrace{\exp_{q^{-1}}\left(\frac{k_+}{\epsilon_-}f_0/(q-q^{-1})\right)}_{\rightarrow\ \tilde{\mathcal{Z}}_1}|0\rangle_B\ ,\quad |+;1\rangle_B = \underbrace{\exp_q\left(-\frac{k_+}{\epsilon_+}e_1q^{-h_1}/(q-q^{-1})\right)}_{\rightarrow\ \tilde{\mathcal{Z}}_1}|1\rangle_B .$$

The complete solution to the spectral problem is given by

$$t^{(i)}(\zeta)\ \underbrace{\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)}_{TypeII\ q-VO}|+;i\rangle_B = \underbrace{\Lambda^{(i)}(\zeta;\xi_1,...,\xi_m;r)}_{\text{identical to the diagonal case}}\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|+;i\rangle_B$$

Application I : generic boundary conditions (in prep.)

The **Hamiltonian of the half-infinite XXZ** spin chain with generic integrable boundary conditions is given by :

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\Big(\sigma_1^{k+1}\sigma_1^k+\sigma_2^{k+1}\sigma_2^k+\Delta\sigma_3^{k+1}\sigma_3^k\Big)-\frac{(q-q^{-1})}{4}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_3^1-\frac{(k_+\sigma_+^1+k_-\sigma_-^1)}{(\epsilon_++\epsilon_-)}$$

In **Onsager's framework**, the **transfer matrix** is :

$$t^{(i)}(\zeta) = g\frac{(\zeta^2-\zeta^{-2})}{\rho(\zeta)}\Big(\epsilon_+\mathcal{W}_-^{(i)}(\zeta)+\epsilon_-\mathcal{W}_+^{(i)}(\zeta)+\frac{1}{q^2-q^{-2}}(k_-\mathcal{Z}_-^{(i)}(\zeta)+k_+\mathcal{Z}_+^{(i)}(\zeta))\Big).$$

STRATEGY : Eigenstates are written as (with S. Belliard, in prep.) :

$$\sum\begin{pmatrix}\mathsf{K}_0^n\mathsf{K}_1^m\mathcal{Z}_1^{\lambda_0}\tilde{\mathcal{Z}}_1^{\lambda_1}\dots\mathcal{Z}_1^{\lambda_{r(even)}}\\ \mathsf{K}_0^n\mathsf{K}_1^m\mathcal{Z}_1^{\lambda_0}\tilde{\mathcal{Z}}_1^{\lambda_1}\dots\tilde{\mathcal{Z}}_1^{\lambda_{r(odd)}}\end{pmatrix}|B_\pm\rangle \quad \text{iff} \quad \lambda_0<\lambda_1<\dots<\lambda_i\geq\lambda_{i+1}\geq\dots\geq\lambda_r$$

The coefficients are determined in terms of **new parameters** α_1,α_2 which are related with the **boundary parameters** $\epsilon_\pm = e^{\pm\xi}, k_\pm = k\frac{(q-q^{-1})}{2}e^{\pm\alpha_0}$:

$$\cosh^2\alpha_1\cosh^2\alpha_2=\frac{\cosh^2\xi}{k^2} \quad \text{and} \quad \cosh^2\alpha_1+\cosh^2\alpha_2=1+\frac{1}{k^2}$$

REMARK : analog to boundary SG (Ghoshal-Zamolodchikov, 1994)

Application II : correlation functions and form factors (beyond the diagonal case)

- The **fundamental eigenstates** are written as **monomials** in the augmented $q-$Onsager algebra generators $\mathsf{Z}_1, \tilde{\mathsf{Z}}_1$ **acting** on the states $|0\rangle_B$, $|1\rangle_B$.

- The **generators** $\mathsf{Z}_1, \tilde{\mathsf{Z}}_1$ are realized as :

$$\mathsf{Z}_1 = (q^2 - q^{-2})\big(\bar{\epsilon}_+ q^{-1} e_0 q^{h_1} + \bar{\epsilon}_- f_1 q^{h_1+h_0}\big) ,$$
$$\tilde{\mathsf{Z}}_1 = (q^2 - q^{-2})\big(\bar{\epsilon}_- q^{-1} e_1 q^{h_0} + \bar{\epsilon}_+ f_0 q^{h_1+h_0}\big)$$

$\rightarrow$ in terms of Drinfeld generators $x_k^{\pm}$ of $U_q(\widehat{sl_2})$

- **Scalar products** between fundamental eigenstates and dual ones are easily computed : $\exp_q(x)\exp_{q^{-1}}(-x) = 1$

The **formalism of** $q-$**vertex operators** (Jimbo et al.) can be used in a straightforward manner in order to derive correlation functions and form factors :

$\Rightarrow$ **Result 1 : Integral representations for the general correlation function**

$${}_B\langle \pm; i| \underbrace{\Phi_{\epsilon_1}^{(i,1-i)}(\zeta_1)\cdot\Phi_{\epsilon_{2M}}^{(1-i,i)}(\zeta_{2M})}_{TypeI\ \rightarrow\ \sigma_a^k}\underbrace{\Psi_{\mu_1}^{*(i,1-i)}(\xi_1)\cdot\Psi_{\mu_{2N}}^{*(1-i,i)}(\xi_{2N})}_{TypeII\ \rightarrow\ excited\ states}|\pm; i\rangle_B$$

$\Rightarrow$ **Result 2 : Linear relations between mutliple integrals** (triangular bcs.)

Identities for integrals of $\Theta_p(z) = (z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty$

Example : correlation functions for a triangular boundary (new)

For a **triangular boundary** $\epsilon_\pm \neq 0, k_+ = 0$, the **Hamiltonian** reads $(r = -\epsilon_+/\epsilon_-)$:

$$H = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) + \frac{q-q^{-1}}{4}\frac{(1+r)}{(1-r)}\sigma_3^1 + \frac{k_-}{(1-r)}\sigma_-^1$$

$\longrightarrow$ Up to now, no results available for correlations functions or form factors.

Non-vanishing correlation functions : $K = \frac{1}{2}(M-N) - |A| + |B| \geq 0$

$$\frac{{}_B\langle i;-|\Phi_{\epsilon_1}^{(i,i+1)}(\zeta_1)\cdots\Phi_{\epsilon_M}^{(i+M-1,i+M)}(\zeta_M)\Psi_{\mu_1}^{*(i+N,i+N-1)}(\xi_1)\cdots\Psi_{\mu_N}^{*(i+1,i)}(\xi_N)|-;i\rangle_B}{{}_B\langle i;-|-;i\rangle_B} \sim$$

$$k_-^K \sum_{l+m=K} \frac{(-1)^{m(M+1)}q^{(l(l-1)-m(m-1))/2(2i-1)-3mM}}{[l]_q![m]_q!} \oint \dots \oint_{C_l^{(i;-)}} (\dots)$$

Remark : for $K=0$, the solution coincides with the diagonal case $(k_\pm = 0)$.

Ex :

$$\frac{{}_B\langle 0;-|\sigma_+^1|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = -\frac{k_-}{\epsilon_-(q-q^{-1})}\left(2 + (1-r)\sum_{l=1}^{\infty}(-q^2)^l\frac{2q^{2l}-r(1+q^{4l})}{(1-rq^{2l})^2}\right),$$

$$\frac{{}_B\langle 0;-|\sigma_z^1|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = -1 - 2(1-r)^2\sum_{l=0}^{\infty}\frac{(-q^2)^l}{(1-rq^{2l})^2} \ ,$$

$$\frac{{}_B\langle 0;-|\sigma_-^1|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = 0 \ .$$

Special cases : $k_- = 0$, or $r=-1$ (free), or $r=1$ (fixed), or $U_q(sl_2)$ inv. $(r = 0, \infty)$.

Application II : Identities involving Theta functions

Correlation functions for the model with $k_+ \neq 0, k_- = 0$ are also computed following the $q-$vertex operator approach. According to the **spin reversal symmetry** (relating the Hamiltonian $\mathcal{H}_{k_-=0} \rightarrow \mathcal{H}_{k_+=0}$, one has for instance ($r = -\epsilon_+/\epsilon_-$) :

$$\underbrace{\frac{{}_B\langle 0; +|\Phi_+(-q^{-1}\zeta)\Phi_+(\zeta)|+;0\rangle_B}{{}_B\langle 0;+|+;0\rangle_B}}_{3\ integrals} = \underbrace{\frac{{}_B\langle 1; -|\Phi_-(-q^{-1}\zeta)\Phi_-(\zeta)|-;1\rangle_B}{{}_B\langle 1;-|-;1\rangle_B}}_{1\ integral}|_{\epsilon_-\rightarrow\epsilon_+, k_-\rightarrow -k_+}$$

More generally, $\exists$ relations between integral representations of different order

Ex : $$2 + \frac{1-z/r}{z}\sum_{k=1}^{\infty}(-q^2)^k \frac{(z-z^{-1}) - (1+q^{4k})/r + (z+z^{-1})q^{2k}}{(1-q^{2k}z/r)(1-q^{2k}/rz)}$$

$$= \frac{(q^2;q^2)^8_\infty}{(q^4;q^4)^4_\infty}\frac{\Theta_{q^4}(z^2)}{1-z^2}\left(-\iiint_{C_0^{(+,1)}} + q^2\iiint_{C_1^{(+,1)}}\right)\prod_{a=1}^{3}\frac{dw_a}{2\pi\sqrt{-1}}\frac{\prod_{a=1}^2(1-q^2/zw_a)}{\prod_{a=1}^2(1-q^2/rw_a)}$$

$$\times\frac{(1-1/rz)(1-q/rw_3)\Theta_{q^2}(w_1w_2)\Theta_{q^2}(w_2/w_1)}{w_1w_2^2w_3^3(1-q^2w_1/w_2)(1-q^4/w_1w_2)}\frac{\Theta_{q^2}(zw_3/q)\Theta_{q^2}(qw_3/z)\prod_{a=1}^{3}\Theta_{q^4}(w_a^2/q^2)}{\prod_{a=1}^{2}\Theta_{q^2}(w_aw_3/q^2)\Theta_{q^2}(w_a/qw_3)\Theta_{q^2}(w_az)\Theta_{q^2}(w_a/z)}.$$

A summary

To **solve** the half-infinite XXZ open spin chain (spectrum, eigenstates, correlation functions) using a combination of Onsager's and $q-$vertex operators approach , successively we have considered :

Step 1 $\rightarrow$ Formulate the **finite XXZ chain** within Onsager's framework for **any type of boundary conditions**. Identify the **spectrum generating algebra** ($q-$Onsager algebra).

Step 2 $\rightarrow$ Thermodynamic limit. Express the **transfer matrix** in terms of the **current algebra** $O_q(\widehat{sl_2})$. Identify **basic modes** and $U_q(\widehat{sl_2})$ realizations.

Step 3 $\rightarrow$ The **central extension** of the reflection equation algebra and $O_q(s\hat{l}_2)$ **algebra** (analog of Miki's formula). Construct **level one infinite dimensional representations** (q-vertex operators) $\rightarrow$ link with $U_q(\widehat{sl_2})\ q-VO$

Step 4 $\rightarrow$ The **spectral problem** for $O_q(s\hat{l}_2)$ currents. Solve the **spectral problem** for the transfer matrix (diagonal/triangular/generic). Eigenstates are constructed based on the linear basis (Ito-Terwilliger) of the $q-$Onsager algebra.

Step 5 $\rightarrow$ **Apply $q-$VO (Kyoto group) formalism** for correlation functions and form factors. Multiple integral representations + Identities for $\Theta_p(z)$ functions.

Perspectives and open problems

- **Short term :**
  - ▷ **Open XXZ at q root of 1** in Onsager's framework :
    $\rightarrow$ *analog of Luztig's higher $q-$Serre relations for the $q-$Onsager algebra required (in preparation, with T.T. Vu)*

  Rem : $XXZ_{q^n=1}$: sl_2 loop symmetry *(Deguchi-Fabricius-McCoy, 2001)*
  open $XXZ_{q^n=1}$: Onsager symmetry expected !

  - ▷ **Local operators** $\sim q-$Onsager Poincaré-Birkhoff-Witt basis
    $\sigma_a^k \sim \sum$ combinations of $O_q(\widehat{sl_2})$ currents $\rightarrow$ OK
    Introducing Davies' relations $\sum \alpha_k \mathcal{W}_k^{(N)} = 0 \rightarrow$ finite case ?

- **Mid term :**
  - ▷ **Analog of Davies' solution** for $q-$Onsager case
    Related work : braid group action (in preparation, with S. Kolb)
  - ▷ **Correlation functions** $\sim$ **Gen. of AW special functions**

  $W_0, W_1 \longleftrightarrow$ *AW 2^{nd} order $q-$diff. op. (Ismail-Lin-Roan,2004)*

Add. 1 : Fundamental elements of $q-$Onsager algebra (finite irrep.)

According to the **choice of boundary conditions** (non-diagonal generic/diagonal), the spectrum generating algebras $\mathcal{A}_q$ ($q-$Onsager algebra) or $\mathcal{A}_q^{diag}$ (augmented $q-$Onsager algebra) arise.

• The **$q-$Onsager algebra :** The **fundamental generators** take the form :

$$\mathcal{W}_0^{(N)} = (k_+\sigma_+ + k_-\sigma_-) \otimes I\!I^{(N-1)} + q^{\sigma_3} \otimes \mathcal{W}_0^{(N-1)} , \quad \mathcal{W}_0^{(0)} = \epsilon_+ ,$$
$$\mathcal{W}_1^{(N)} = (k_+\sigma_+ + k_-\sigma_-) \otimes I\!I^{(N-1)} + q^{-\sigma_3} \otimes \mathcal{W}_1^{(N-1)} , \quad \mathcal{W}_1^{(0)} = \epsilon_- .$$

$\longrightarrow$ $q-$analogs of A_0, A_1 which generate the Onsager algebra (*Onsager, 1944*)
$\longrightarrow$ $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ are explicit examples of tridiagonal pairs (*Def : Terwilliger, 2003*)

• The **augmented $q-$Onsager algebra :** The **fundamental generators** take the form :

$$\mathcal{K}_0^{(N)} = q^{\sigma_3} \otimes \mathcal{K}_0^{(N-1)} , \quad \mathcal{K}_0^{(0)} = \epsilon_+ , \quad \mathcal{K}_1^{(N)} = q^{-\sigma_3} \otimes \mathcal{K}_1^{(N-1)} , \quad \mathcal{K}_1^{(0)} = \epsilon_- ,$$
$$\mathcal{Z}_1^{(N)} = I\!I \otimes \mathcal{Z}_1^{(N-1)} + (q^2 - q^{-2})\sigma_- \otimes \left(\mathcal{K}_0^{(N-1)} + \mathcal{K}_1^{(N-1)}\right) ,$$
$$\tilde{\mathcal{Z}}_1^{(N)} = I\!I \otimes \tilde{\mathcal{Z}}_1^{(N-1)} + (q^2 - q^{-2})\sigma_+ \otimes \left(\mathcal{K}_0^{(N-1)} + \mathcal{K}_1^{(N-1)}\right) , \quad \mathcal{Z}_1^{(0)} = \tilde{\mathcal{Z}}_1^{(0)} = 0 .$$

$\longrightarrow$ Definition of the augmented q-Onsager algebra : *Ito-Terwilliger, 2010*
$\longrightarrow$ $q-$analog of the augmented Onsager algebra (*Belliard-Crampé, 2012*)

Add 2 : Hidden symmetry and special points

A. HIDDEN SYMMETRY : For the infinite XXZ spin chain, the Hamiltonian enjoys the $U_q(\widehat{sl_2})$ non-Abelian symmetry (Jimbo et al.,1993). For the half-infinite XXZ open chain, from the thermodynamic limit of the finite chain emerge :

- For non-diagonal boundary parameters $k_\pm \neq 0$: the $q-$**Onsager algebra**

$$[H, a] = 0 \,, \qquad a \in \{\mathcal{W}_0^{(\infty)}, \mathcal{W}_1^{(\infty)}\} \,.$$

- For diagonal boundary parameters $k_\pm = 0$ the **augmented** $q-$**Onsager algebra**

$$[H^{diag}, a] = 0 \,, \qquad a \in \{\mathcal{K}_0^{(\infty)}, \mathcal{K}_1^{(\infty)}, \mathcal{Z}_1^{(\infty)}, \tilde{\mathcal{Z}}_1^{(\infty)}\} \,.$$

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B. SPECIAL POINTS : $\epsilon_+ = 0, \epsilon_- \neq 0$ (resp. $\epsilon_- = 0, \epsilon_+ \neq 0$), $k_\pm = 0$

$$H^{(\pm)} \quad = \quad -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) \pm \frac{(q-q^{-1})}{4}\sigma_3^1$$

$H^{(\pm)} \sim$ discrete analog of the Virasoro generator L_0 (Pasquier-Saleur, 1990).

$\rightarrow$ **Spectrum :** $H^{(+)} \sim \mathrm{W}_0$ and $H^{(-)} \sim \mathrm{W}_1$ ($q-$Onsager) simple structure
$\rightarrow$ **Eigenstates :** from $|B_+\rangle$ (resp. $|B_-\rangle$) using type II vertex op.
$\rightarrow$ **Correlation functions :** $\sim$ Generalizations of Askey-Wilson polynomials

Relation with $q-$Virasoro ?