Correlation functions in algebraic Bethe Ansatz framework and Drinfel'd second realization of quantum groups

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Periodic quantum integrables models in RLL framework [Leningrad 80s] :

$$R_{ab}(u/v)L_a(u)L_b(v) = L_b(v)L_a(u)R_{ab}(u/v),$$

$$L(u) = \sum_{ij} L_{ij}(u) \otimes E_{ij}, \quad \Delta(L(u)) = \sum_{ij} L_{ik}(u) \otimes L_{kj}(u) \otimes E_{ij},$$

Transfer matrix : $t(u) = tr_a(L(u)), \quad [t(u), t(v)] = 0$

$\Rightarrow$ Generating function of quantum integrable Hamiltonians : $\frac{d}{du} ln(t(u)) \sim H$

• We consider, for Quantum group $\mathcal{U}_q(\widehat{sl(n)})$ and related models, problems :

A - Spectrum and states of H : $H|\phi\rangle = E|\phi\rangle$

B - Correlation functions of local operators $\mathcal{O}$: $F_{\mathcal{O}} = \langle\phi|\mathcal{O}|\phi\rangle$

$\Rightarrow$ Different approaches : BA, SoV, Functional,...

• Here we focus on models that can be solve by the **algebraic Bethe ansatz** !

Reference vector :

$$|0\rangle,\ L_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle,\ L_{ij}(u)|0\rangle = 0 \quad \text{for} \quad i > j$$

Bethe Vectors :

$$|\phi^a(\bar{u})\rangle = V^a(L_{ij}(\bar{u}))|0\rangle,\ 1 < i \leq j < n,$$

V^a a polynomial and $\{\bar{u}\}$ set of Bethe roots (divided on $n-1$ subset)

- OK for sl_2 and for sl_n with different formulations
  - **with** auxiliary space : sl_n, [Kulish-Reshetikhin 83 ; Tarasov-Varchenco 95]
  - **without** auxiliary space : sl_3 [B-P-R-S 12-13], sl_n case ?

Action of the transfer matrix on the Bethe Vectors :

$$t(w)|\phi^a(\bar{u})\rangle = \underbrace{\tau(w,\bar{u})|\phi^a(\bar{u})\rangle}_{WT} + \overbrace{\sum_i G(w,u_i)B_i^a(\bar{u})(|\phi^a(\bar{u}_i,w)\rangle + \dots)}^{UWT},$$

Imposing **Bethe equations** : $B_i^a(\bar{u}) = 0 \quad \Rightarrow \quad |\phi^a(\bar{u})\rangle$ is On-shell

- OK for sl_n [Kulish-Reshetikhin 83] (coordinate BA [Sutherland 75])

  $\Rightarrow$ **explicit unwanted terms** (UWT) for sl_3 [B-P-R-S 12-13], sl_n case ?

Reconstruction of $\mathcal{O}$ from $\mathcal{U}_q(\widehat{sl_n})$: $\mathcal{O}(\bar{w}) = \mathcal{IP}(L_{ij}(\bar{w}))$

- OK for sl_2 and for sl_n (fondamental representation) [Maillet-Terras 2000]

  $\Rightarrow sl_n$ non-fondamental representations ?

Action of the entries of monodromy matrix on Bethe vectors

$$L_{ij}(\bar{w})|\phi^a(\bar{u})\rangle = \sum_{\{\bar{u},\bar{w}\}\to\{\bar{x}^{(1)},\bar{x}^{(2)}\}} G_{ij}(\bar{u},\bar{w},\bar{x}^{(1)})|\phi^{a'}(\bar{x}^{(2)})\rangle$$

- OK for sl_2 and for sl_3 [Leningrad 80s], [B-P-R-S 12-13]

  $\Rightarrow sl_n$ case ?

Correlation functions of $\mathcal{O}$ $\Rightarrow$ **Scalar product** of **On-shell** and **Off-shell** BV

$$\langle\phi^b(\bar{v})|\mathcal{O}(\bar{w})|\phi^a(\bar{u})\rangle = \sum_{\{\bar{u},\bar{w}\}\to\{\bar{x}^{(1)},\bar{x}^{(2)}\}} W_{\mathcal{O}}(\bar{u},\bar{w},\bar{x}^{(1)})\langle\phi^b(\bar{v})|\phi^{a'}(\bar{x}^{(2)})\rangle$$

- Determinant form OK for sl_2 and for sl_3 (XXX some cases) [Korepin, Izergin, Slavnov 80s], [Reshetikhin 86 ; Wheeler 12-13], [B-P-R-S 12]

  $\Rightarrow sl_3$ XXZ, sl_n case ?

Drinfel'd currents [Drinfel'd 88] : $\{e_i(u), f_i(u), \psi_i^{\pm}(u), (\gamma = 1)\}$, $i = 1, \dots, n-1$

$$\psi_i^{\pm}(u)\psi_j^{\pm}(v) = \psi_j^{\pm}(v)\psi_i^{\pm}(u), \quad \psi_i^{\pm}(u)f_j(v) = \frac{q^{a_{ij}}u - v}{u - vq^{a_{ij}}} f_j(v)\psi_i^{\pm}(u),$$

$$f_i f_j(v) = \frac{u - q^{a_{ij}}v}{q^{a_{ij}}u - v} f_j(v) f_i(u), \quad [e_i(u), f_j(v)] = \frac{\delta_{ij}\delta(u/v)}{(q - q^{-1})}(\psi_i^{-}(u) - \psi_i^{+}(v)),$$

with other relations and $A = \{a_{ij}\}$ Cartan matrix of sl_n

- **Remark** : $\mathcal{U}_q(\widehat{sl_n})$ (RLL realization)

$\rightarrow \{L^{\pm}(u), (\gamma = 1)\}$ [Reshetikhin-Semenov 89].

embedding : $L(u) \rightarrow L^{+}(u) \in \mathcal{U}_q(\widehat{b}_+)$ the positive Borel subalgebra of $\mathcal{U}_q(\widehat{sl_n})$

Half currents and "Gauss" decomposition [Ding-Frenkel 93]

$$L^{\pm}(u) = F^{\pm}(u)H^{\pm}(u)E^{\pm}(u), \quad H^{\pm}(u) = \sum k_i^{\pm}(u)E_{ii}$$

Upper and lower triangular matrices $\left\{ \begin{array}{l} E^{\pm}(u) = 1 + \sum e_i^{\pm}(u)E_{ii+1} + \dots \\ F^{\pm}(u) = 1 + \sum f_i^{\pm}(u)E_{i+1i} + \dots \end{array} \right.$

Full currents $\left\{ \begin{array}{l} e_i(u) = e_i^{+}(u) - e_i^{-}(u) \\ f_i(u) = f_i^{+}(u) - f_i^{-}(u) \end{array} \right.$, $\psi_i^{\pm}(u) = (k_{i+1}^{\pm}(u)k_i^{\pm}(q^{-2}u))^{-1}$

Projector : intersection of Borel subalgebras $\{f_i(u), k_i^+(u)\} \bigcap \{L^+(u)\}$

$$f_i^+(u) = P^+(f_i(u)) = \oint_\infty \frac{dx}{x} \frac{u/x}{1-u/x} f(x),$$

Reference vector : $\qquad k_i^\pm(u)|0\rangle = \lambda_i^\pm(u)|0\rangle, \quad e_i(u)|0\rangle = 0$

Bethe vectors and **partition functions** of $2D$ vertex models

$$|\phi^a(\bar{u})\rangle \sim P^+(f(\bar{u}))|0\rangle \sim \oint_\infty \frac{d\bar{x}}{\bar{x}} \overbrace{\mathcal{Z}^{(2)}(\bar{x}|\bar{u})}^{Izergin\,PF} f(\bar{x})|0\rangle, \quad f(\bar{u}) = f(u_1)\dots f(u_a)$$

$$|\phi^{ab}(\bar{u},\bar{v})\rangle \sim P^+(f_1(\bar{u})f_2(\bar{v}))|0\rangle \sim \oint_\infty \frac{d\bar{x}}{\bar{x}} \frac{d\bar{y}}{\bar{y}} \overbrace{\mathcal{Z}^{(3)}(\bar{x},\bar{y}|\bar{u},\bar{v})}^{Reshetikhin\,PF} f_2(\bar{y}) f_1(\bar{x})$$

- Applications for problem **A-B** :
  - **Bethe vectors** : integral formulation, explicit formulation (isomorphism RLL)
  - **Action of $L_{ij}(u)$ on Bethe vectors**, **Scalar product** (integral formulation)

$\Rightarrow$ Similar formulation for Bethe Vectors of open models ?

$\Rightarrow$ Coideal subalgebras of Drinfel'd Currents ?

Open quantum integrable models in reflection algebra framework [Sklyanin 88]

$$R_{12}(u/v)K_1(u)R_{21}(u\,v)K_2(v) = K_2(v)R_{12}(u\,v)K_1(u)R_{21}(u/v)$$

$$K(u) = \big(L(\alpha\,u)\big)^{-1}K_s(u)L(\alpha\,u^{-1})$$

$$\delta(K(u)) = \sum_{ij} L'_{ik}(\alpha\,u)L_{lj}(\alpha\,u^{-1}) \otimes K_{kl}(u) \otimes E_{ij}$$

Transfer matrix : $d(u) = tr(\bar{K}_s(u)K(u)), \quad [d(u), d(v)] = 0$

$\Rightarrow$ Generating function of open quantum integrable Hamiltonians : $\frac{d}{du}ln\big(d(u)\big) \sim H$

- ABA (no-gauge transformation) :
  - diagonal boundaries sl_2 case [Sklyanin 88] (triangular XXX [B-C-R 12], XXZ [Pimenta-Lima Santos 13])
  - diagonal boundaries sl_n case [de Vega-Gonzalez Ruiz 93], trace formula [B-R 09]
- Correlation functions from ABA :
  - diagonal boundaries sl_2 case [Lyon group 09]

RLL / RKRK
$\{L^{\pm}(u)\}$ / $\{K^{\epsilon_1,\epsilon_2}(u)\}$
ABA/SoV/Functional

⟷

Drinfel'd realization / ? ?
$\{e_i(u), f_i(u), \psi_i^{\pm}(u)\}\ i = 1, \dots n$
bosonization/vertex operators
Bethe vectors by projection

$\neq$ **realization of $\mathcal{U}_q(\widehat{sl_2})$/Coideals**

q-Onsager Currents
$\{\mathcal{W}_{\pm}(u), \mathcal{Z}_{\pm}(u)\}$
Onsager approach

Drinfeld-Jimbo / type q-Onsager
$\{e_i, f_i, h_i\}$ / $\{A_i\}\ i = 0, \dots n$
non-local conserved quantities

- Realizations of $\mathcal{U}_q(\widehat{sl_2})$, isomorphisms OK [Jimbo, Drinfel'd, Ding-Frenkel... 80-90s]
- Realizations of Coideals of $\mathcal{U}_q(\widehat{sl_2})$
  - **RKRK OK** : $\{L^{-1}(u^{-1}),\ L^t(u^{-1}),\ K_s(u)\}$ [Cherednick, Sklyanin 80s ; Molev-Ragoucy, Guay, ... 00-10s]
  - **Type $\mathcal{O}_q$ OK** : (augmented) q-Onsager [B-B 09-12][B-C 12], $\mathcal{U}_q(\widehat{sl_2})$-invariant q-Onsager [MacKay-Delius, Baseilhac, Terwilliger, Regelskis, Kolb, ... 00-10s]
  - **$\mathcal{O}_q$ currents OK** : [Baseilhac-Shigechi 09] [B-B 12], others coideals $\widehat{g}$?

How to construct coideal subalgebras ? **automorphism + (Drinfel'd) coproduct**

$$\theta(e(qu)) = u^{-1}e(qu^{-1}), \qquad \Delta_D(e(u)) = e(u)\otimes 1 + \psi^-(u)\otimes e(u)$$
$$\theta(f(qu)) = uf(qu^{-1}), \qquad \Delta_D(f(u)) = f(u)\otimes\psi^+(u) + 1\otimes f(u),$$
$$\theta(\psi^+(qu)) = \left(\psi^-(qu^{-1})\right)^{-1}, \qquad \Delta_D(\psi^\pm(u)) = \psi^\pm(u)\otimes\psi^\pm(u)$$

- **Classical limit** : Augmented Onsager algebra $\{K(u), B(u), \bar{B}(u)\}$ [B-C 12]
- **Coideal property** :

$$\delta_D \sim \Delta_D\circ\phi_q : \mathcal{AO}_q^{DR}(\widehat{sl_2}) \to \mathcal{U}_q^{DR}(\widehat{sl_2})\otimes\mathcal{AO}_q^{DR}(\widehat{sl_2})$$

$\mathcal{AO}_q^{DR}(\widehat{sl_2})$ currents

Expected currents

$$\begin{cases} \phi_q(\mathcal{K}(u)) = (\psi^+(qu))^{-1}\psi^-(qu^{-1}), \\ \phi_q(\mathcal{B}(u)) = e(qu^{-1}) + e(qu)(\psi^-(qu))^{-1}\psi^-(qu^{-1}), \\ \phi_q(\bar{\mathcal{B}}(u)) = (f(qu)+f(qu^{-1}))(\psi^+(qu))^{-1}, \end{cases}$$

Aditional currents

$$\begin{cases} \phi_q(\mathcal{C}^\pm(u)) = (\psi^\pm(qu))^{-1}\psi^\pm(qu^{-1}), \\ \phi_q(\bar{\mathcal{C}}(u)) = (\psi^-(qu))^{-1}\psi^+(qu). \end{cases}$$

$\mathcal{AO}_q^{DR}(\widehat{sl_2})$ is generated by $\{\mathcal{B}(u), \bar{\mathcal{B}}(u), \mathcal{K}(u), \mathcal{C}^{\pm}(u), \bar{\mathcal{C}}(u), (\gamma = 1)\}$

$$\mathcal{K}(u)\mathcal{K}(v) = \mathcal{K}(v)\mathcal{K}(u),$$

$$\mathcal{K}(u)\mathcal{B}(v) = \frac{(q^{-2}uv-1)(q^{-2}u-v)}{(uv-q^{-2})(u-q^{-2}v)}\mathcal{B}(v)\mathcal{K}(u),$$

$$\mathcal{K}(u)\bar{\mathcal{B}}(v) = \frac{(q^{2}uv-1)(q^{2}u-v)}{(uv-q^{2})(u-q^{2}v)}\bar{\mathcal{B}}(v)\mathcal{K}(u),$$

$$\mathcal{B}(u)\mathcal{B}(v) = \frac{(q^{-2}u-v)}{(u-q^{-2}v)}\mathcal{B}(v)\mathcal{B}(u), \quad \bar{\mathcal{B}}(u)\bar{\mathcal{B}}(v) = \frac{(q^{2}u-v)}{(u-q^{2}v)}\bar{\mathcal{B}}(v)\bar{\mathcal{B}}(u),$$

$$\mathcal{B}(u)\bar{\mathcal{B}}(v) = \frac{q^2uv-1}{uv-q^2}\bar{\mathcal{B}}(v)\mathcal{B}(u)$$
$$+\frac{\delta(uv)}{q-q^{-1}}\Big(1-\bar{\mathcal{C}}(v)-\frac{(q^2uv-1)(q^2u-v)}{(uv-q^2)(u-q^2v)}(\mathcal{K}^{-1}(v)\mathcal{K}(u)-\bar{\mathcal{C}}(v))\Big)$$
$$+\frac{\delta(u/v)}{q-q^{-1}}\Big(\mathcal{C}^{+}(u)-\mathcal{K}(u)+\frac{(q^2uv-1)(q^2u-v)}{(uv-q^2)(u-q^2v)}(\mathcal{C}^{-}(v)-\mathcal{K}(v))\Big)$$

and other relations ...

Use the Gauss decomposition and Sklyanin dressing procedure

$$K^{-+}(qu) = \big(L^-(q^3u)\big)^{-1} K_{sd}(qu) L^+(qu^{-1}), \quad K_{sd}(u) = \begin{pmatrix} a(u) & 0 \\ 0 & a(u^{-1}) \end{pmatrix}$$

As example, ordering the gauss coordinates, $K^{-+}_{12}(u)$ can be write as

$$K^{-+}_{12}(qu) \sim \bar{\mathcal{B}}^{-+}(u)\bar{\mathcal{C}}^{-+}(u)\mathcal{W}^+(u)$$

with $\quad \bar{\mathcal{B}}^{-+}(u) = \Big(a(qu)f^-(qu) - a(qu^{-1})f^+(qu^{-1})\Big)(\psi^+(qu))^{-1}$

and $\quad \mathcal{W}^+(u) = k^+(qu^{-1})\big(k^+(q^{-1}u)\big)^{-1}, \quad \bar{\mathcal{C}}^{-+}(u) = k^-(qu)\big(k^+(qu)\big)^{-1}$

Cartan currents are in $\mathcal{AO}^{DR}(\widehat{sl_2})$: $\{\mathcal{C}^\pm(u), \bar{\mathcal{C}}(u)\} \to \{\mathcal{W}^\pm(u), \bar{\mathcal{C}}^{-+}(u)\}$

- For $\mathcal{U}_q(\widehat{sl_2})$: $\psi^\pm(u) = (k^\pm(u)k^\pm(q^{-2}u))^{-1}$
- For $\mathcal{AO}^{DR}_q(\widehat{sl_2})$: $\mathcal{C}^\pm(u) = \big(\mathcal{W}^\pm(u)\mathcal{W}^\pm(q^2u)\big)^{-1}, \quad \bar{\mathcal{C}}(u) = \bar{\mathcal{C}}^{-+}(u)\bar{\mathcal{C}}^{-+}(q^{-2}u)$

$\Rightarrow$ **Relations close** for $\{\mathcal{K}^{-+}(u), \mathcal{W}^\pm(u), \bar{\mathcal{C}}^{-+}(u)\}$

$$\mathcal{W}^+(u)\bar{\mathcal{B}}(v) = \frac{(q^2-uv)(q^2-u/v)}{(1-uv)(1-u/v)}\bar{\mathcal{B}}(v)\mathcal{W}^+(u), \quad \bar{\mathcal{C}}^{-+}(u)\bar{\mathcal{B}}(v) = \bar{\mathcal{B}}(v)\bar{\mathcal{C}}^{-+}(u),$$

- **Actual projects** :
  - Drinfel'd realization of Coideals of $\mathcal{U}_q(\widehat{sl_2})$ [B, in preparation]
  - Bethe vectors of open models in Drinfel'd realization ($sl(2)$ case)
- **Open questions** :
  - Drinfel'd realization of coideals of $\mathcal{U}_q(\widehat{sl(3)})$ and related Bethe vectors.

Scalar products, action of $L_{ij}(u)$ in Drinfel'd realization [1012.1455, 1304.7602]
with S. Pakuliak, E. Ragoucy, N. A. Slavnov

ABA and Open spin Chain [0902.0321, 1209.4269]
with N. Crampé, É. Ragoucy

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q-Onsager, reflection algebra, XXZ [0906.1215, 1104.1591, 1211.6304]
with P. Baseilhac

deformation of symmetric spaces [1202.2312]
with N. Crampé

Thanks you for attention !