Correlation functions of the SOS model from algebraic Bethe Ansatz

Finite-size results

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- "An algebraic Bethe ansatz approach to form factors and correlation functions of the cyclic eight-vertex solid-on-solid model" J. Stat. Mech. (2013) P04015, `arXiv:1212.0246`
- "Spontaneous staggered polarizations of the cyclic SOS model from algebraic Bethe Ansatz" `arXiv:1304.7814`
- "Multi-point local height probabilities of the CSOS model within the algebraic Bethe Ansatz framework", in preparation

Algebraic Bethe ansatz approach to correlation functions

1. **Correlation functions in the ABA framework: first results**

   determinant representation for scalar products of Bethe states (Slavnov)

   + solution of the quantum inverse problem

   $\rightsquigarrow$ determinant representation for form factors in finite volume

   $\rightsquigarrow$ elementary building blocks of correlation functions as multiple sums in finite volume and as multiple integrals in the thermodynamic limit

2. **Two-point function:** sum up elementary blocks or form factors

   $\rightsquigarrow$ Master equation representation for the finite chain: N-fold multiple integral representation for the correlation function in finite volume

3. **Asymptotic analysis of the two-point function**

   $\rightsquigarrow$ from the Master equation

   $\rightsquigarrow$ from the series over form factors

Method essentially developed for XXZ chain or Quantum Bose gas

What about more complicated models ?

XYZ Heisenberg chain and 8VSOS model

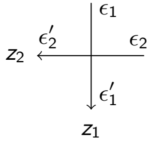
A natural generalization of the XXZ Heisenberg chain is the XYZ chain:

$$H_{XYZ} = \sum_{m=1}^{N} \left\{ J_x \sigma_m^x \sigma_{m+1}^x + J_y \sigma_m^y \sigma_{m+1}^y + J_z \sigma_m^z \sigma_{m+1}^z \right\}$$

related to the 8-vertex model:

2-d square lattice model
link $\longrightarrow \epsilon_j = \pm$
vertex $\longrightarrow$ Boltzmann weight

$$\mathbf{R}^{8V}(z_1/z_2)_{\epsilon_1',\epsilon_2'}^{\epsilon_1,\epsilon_2} =$$



$$\mathbf{R}^{8V}(z) = \begin{pmatrix} a(z;p) & 0 & 0 & d(z;p) \\ 0 & b(z;p) & c(z;p) & 0 \\ 0 & c(z;p) & b(z;p) & 0 \\ d(z;p) & 0 & 0 & a(z;p) \end{pmatrix}$$

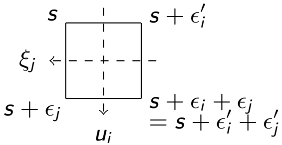
z = spectral parameter
p = elliptic parameter
a, b, c, d = elliptic theta functions of z

No charge conservation through a vertex $\longrightarrow$ no direct Bethe Ansatz solution
Baxter's solution (Ann.Phys.73) $\longrightarrow$ map onto an IRF model (8VSOS model)
eigenstates of 8V model given in terms of Bethe eigenstates of 8VSOS model

8VSOS model

2-d square lattice model
vertex $\longrightarrow$ local height s_j
$s_j - s_k = \pm 1$ (adjacent)
face $\longrightarrow$ Boltzmann weight

$$\mathbf{R}(u_i - \xi_j; s)^{\epsilon_i, \epsilon_j}_{\epsilon'_i, \epsilon'_j} =$$



$$\mathbf{R}(u; s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u;s) & c(u;s) & 0 \\ 0 & c(u;-s) & b(u;-s) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$b(u;s) = \frac{[s+1][u]}{[s][u+1]}$

$c(u;s) = \frac{[s+u][1]}{[s][u+1]}$

$u =$ spectral parameter

$s =$ dynamical parameter

$[u] = \theta_1(\eta u; \tau) \qquad p = e^{2\pi i \tau}$

satisfying the **Dynamical** Quantum Yang-Baxter Equation:

$$\mathbf{R}_{12}(u_1 - u_2; s + h_3)\, \mathbf{R}_{13}(u_1; s)\, \mathbf{R}_{23}(u_2; s + h_1)$$

$$= \mathbf{R}_{23}(u_2; s)\, \mathbf{R}_{13}(u_1; s + h_2)\, \mathbf{R}_{12}(u_1 - u_2; s) \qquad \text{with} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Charge conservation, solvable by Bethe Ansatz

ABA for the 8VSOS model

Felder, Varchenko (1996) : representations of $E_{\tau,\eta}(sl_2)$

- **Monodromy matrix:**

$$T_{a,1\dots N}(u;\xi_1,\dots,\xi_N;s) = R_{aN}(u-\xi_N;s+h_1+\dots+h_{N-1})\dots R_{a1}(u-\xi_1;s)$$
$$= \begin{pmatrix} A(u;s) & B(u;s) \\ C(u;s) & D(u;s) \end{pmatrix}_{[a]} \in \mathrm{End}(\mathbb{C}^2\otimes\mathcal{H})$$

$$\widehat{T}(u) = \begin{pmatrix} \widehat{A}(u) & \widehat{B}(u) \\ \widehat{C}(u) & \widehat{D}(u) \end{pmatrix}_{[a]} = T(u;\widehat{s}) \begin{pmatrix} \widehat{\tau}_s & 0 \\ 0 & \widehat{\tau}_s^{-1} \end{pmatrix}_{[a]} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathrm{Fun}(\mathcal{H})),$$

where $\widehat{\tau}_s\,\widehat{s} = (\widehat{s}+1)\,\widehat{\tau}_s$, and the action of $\widehat{s}$ and $\widehat{\tau}_s$ on functions $f\in\mathrm{Fun}(\mathcal{H})$ are given as $[\widehat{s}f](s) = sf(s)$, $[\widehat{\tau}_s f](s) = f(s+1)$.

- **Transfer matrix:** $\widehat{t}(u) = \widehat{A}(u) + \widehat{D}(u)$

$\rightsquigarrow$ preserve the space $\mathrm{Fun}(\mathcal{H}[0])$ of functions of the dynamical parameter s with values in the zero-weight space $\mathcal{H}[0]$ of $\mathcal{H}$

$\rightsquigarrow$ $[\widehat{t}(u),\widehat{t}(v)] = 0$ on $\mathrm{Fun}(\mathcal{H}[0])$

- **Space of states:** functions $\psi : s \mapsto \psi(s) \in \mathcal{H}[0]$
  - unrestricted case (η generic): $s \in \mathbf{C}_{s_0} = \{s_0 + j, j\in\mathbb{Z}\}$
  - cyclic case ($\eta = r/L$ rational): $s \in \mathbf{C}^{\mathbf{L}}_{s_0} = \{s_0 + j, j \in \mathbb{Z}/L\mathbb{Z}\}$

- **reference state:** $|\,0\,\rangle = \otimes_{j=1}^{N} \binom{1}{0}_{[j]}$

  $A(u;s)|\,0\,\rangle = a(u)|\,0\,\rangle, \qquad D(u;s)|\,0\,\rangle = \frac{[s-1]}{[s+N-1]} d(u)|\,0\,\rangle$

- **Bethe states:** Suppose that the set of spectral parameters $\{v_1,\ldots,v_n\}$, satisfies the system of Bethe equations

$$a(v_j)\prod_{l\neq j}\frac{[v_l-v_j+1]}{[v_l-v_j]} = (-1)^{rk}\omega^{-2}\, d(v_j)\prod_{l\neq j}\frac{[v_j-v_l+1]}{[v_j-v_l]}, \quad j=1,\ldots n,$$

with $N=2n+kL$ ($k\in\mathbb{Z}$) and $\omega^L=(-1)^{rn}$ (for $\eta=r/L$), then the states

$$|\,\{v\},\omega\,\rangle : s\mapsto \varphi_\omega(s)\, B(v_1;s)B(v_2;s-1)\ldots B(v_n;s-n+1)|\,0\,\rangle,$$
$$\langle\,\{v\},\omega\,| : s\mapsto \langle\,0\,|C(v_n;s-n)\ldots C(v_2;s-2)C(v_1;s-1)\,\bar{\varphi}_\omega(s),$$

with $\varphi_\omega(s)=\omega^s\prod_{j=1}^{n}\frac{[1]}{[s-j]}, \qquad \bar{\varphi}_\omega(s)=\omega^{-s}\prod_{j=0}^{n-1}\frac{[s+j]}{[1]},$

are eigenstates of the transfer matrix

$$\big[\widehat{t}(u)\,|\,\{v\},\omega\,\rangle\big](s) = A(u;s)\;|\,\{v\},\omega\,\rangle(s+1) + D(u;s)\;|\,\{v\},\omega\,\rangle(s-1)$$
$$= \tau(u;\{v\},\omega)\;|\,\{v\},\omega\,\rangle(s),$$
$$\big[\langle\,\{v\},\omega\,|\,\widehat{t}(u)\big](s) = \langle\,\{v\},\omega\,|(s-1)\;A(u;s-1) + \langle\,\{v\},\omega\,|(s+1)\;D(u;s+1)$$
$$= \tau(u;\{v\},\omega)\;\langle\,\{v\},\omega\,|(s),$$

with eigenvalue

$$\tau(u;\{v\},\omega) = \omega\; a(u)\prod_{l=1}^{n}\frac{[v_l-u+1]}{[v_l-u]} + (-1)^{rk}\omega^{-1}\; d(u)\prod_{l=1}^{n}\frac{[u-v_l+1]}{[u-v_l]}.$$

Scalar product of Bethe states

Compute $\langle \{u\}, \omega_u | \, \{v\}, \omega_v \rangle$ in a compact and manageable form ?

- **for XXZ:**
  - $\exists$ determinant representation for the scalar product when one of the state is a Bethe eigenstate (Slavnov, 1989)
  - this representation is related to Izergin's determinant representation for the partition function with domain wall boundary conditions:

$$Z_N(\{u\}; \{\xi\}) \propto \det_N \frac{\sinh \eta}{\sinh(u_i - \xi_j) \sinh(u_i - \xi_j + \eta)}$$

- **for SOS:**
  - no single determinant representation for the partition function with DWBC (Rosengren; Pakuliak, Rubtsov, Silantyev)

$$Z_N(\{u\}; \{\xi\}; s) \propto \sum_{S \subset \{1,\ldots,N\}} (-1)^{|S|} \frac{[\gamma + s - |S|]}{[s - |S|]} \det_N \frac{[u_j - \xi_k^S + \gamma]}{[\gamma][u_j - \xi_k^S]}$$

with $\xi_k^S = \begin{cases} \xi_k - 1 & \text{if } \; k \in S \\ \xi_k & \text{if } \; k \notin S \end{cases}$ (γ arbitrary).

Scalar product of Bethe states

Let $\{u\}, \omega_u$ be solution of the Bethe equations and $\{v\}, \omega_v$ be arbitrary, and consider the quantities:

- "partial scalar product" (depending on the value of the height s):

$$S_n(\{u\};\{v\};s) = \langle\, 0\,|\, C(u_n; s-n) \dots C(u_1; s-1) B(v_1; s) \dots B(v_n; s-n+1) |\, 0\,\rangle$$

$\rightsquigarrow$ can be computed from Rosengren's formula for the partition function with DWBC using the expressions of B and C in the F-basis (Maillet, Sanchez de Santos 96; Kitanine,Maillet,V.T. 99; Albert et al. 00)

$\rightsquigarrow$ sum of determinants

- "total scalar product" (cyclic case $\eta = r/L$ rational):

$$\langle \{u\}, \omega_u \,|\, \{v\}, \omega_v \rangle = \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \bar{\varphi}_{\omega_u}(s)\, \varphi_{\omega_v}(s)\, S_n(\{u\};\{v\};s)$$

$\rightsquigarrow$ The "total scalar product" (and the norm) can be expressed as a single determinant

Remark. The partial scalar product is not a scalar product of Bethe-type states (function δ_s instead of φ_{ω_v}, $\bar{\varphi}_{\omega_u}$)

Computation of the partial scalar product (I)

$$S_n(\{u\};\{v\};s) = \langle\, 0\,|\,C(u_n;s-n)\dots C(u_1;s-1)B(v_1;s)\dots B(v_n;s-n+1)|\,0\,\rangle$$
$$= \langle\, 0\,|\,\widetilde{C}(u_n;s-n)\dots \widetilde{C}(u_1;s-1)\widetilde{B}(v_1;s)\dots \widetilde{B}(v_n;s-n+1)|\,0\,\rangle$$

where $\widetilde{C}$ and $\widetilde{B}$ stand for the expressions of C and B in the F-basis (Maillet, Sanchez de Santos 96; Kitanine,Maillet,V.T. 99; Albert et al. 00) :

$$\widetilde{B}(u;s) \equiv F_{1\dots N}(s)\,B(u;s)\,F^{-1}_{1\dots N}(s-1)$$
$$= \frac{[s-1]}{[s+h_{1\dots N}]}\sum_{i=1}^{N}\sigma_i^-\,\frac{[1][s+\sum_{l\neq i}h_l+u-\xi_i]}{[s+\sum_{l\neq i}h_l][u-\xi_i+1]}\bigotimes_{j\neq i}\begin{pmatrix}\frac{[u-\xi_j]}{[u-\xi_j+1]} & 0\\ 0 & \frac{[\xi_j-\xi_i+1]}{[\xi_j-\xi_i]}\end{pmatrix}_{[j]},$$

$$\widetilde{C}(u;s) \equiv F_{1\dots N}(s)\,B(u;s)\,F^{-1}_{1\dots N}(s+1)$$
$$= \sum_{i=1}^{N}\sigma_i^+\,\frac{[1][s-u+\xi_i]}{[s][u-\xi_i+1]}\bigotimes_{j\neq i}\begin{pmatrix}\frac{[u-\xi_j]}{[u-\xi_j+1]}\frac{[\xi_i-\xi_j+1]}{[\xi_i-\xi_j]} & 0\\ 0 & 1\end{pmatrix}_{[j]}.$$

Computation of the partial scalar product (II)

⇝ computable by recursion:

Consider

$$G^{(k)}_{\ell_{k+1},\dots,\ell_n}(\{u\};\{v_1,\dots,v_k\};s)$$
$$= \langle 0\,|\widetilde{C}(u_n;s-n)\dots\widetilde{C}(u_1;s-1)\widetilde{B}(v_1;s)\dots\widetilde{B}(v_k;s-k+1)\;\underbrace{|\,\ell_{k+1},\dots,\ell_n\,\rangle}_{\substack{\text{state with n-k "-" spins}\\ \text{at positions } \ell_{k+1}\dots\ell_n}}\;,$$

such that

- $G^{(0)}_{\ell_1,\dots,\ell_n}(\{u\};\emptyset;s) \propto Z_n(\{u\};\{\xi_{\ell_j}\};s)$
- $G^{(n)}(\{u\};\{v\};s) = S_n(\{u\};\{v\};s)$
- $$G^{(k)}_{\ell_{k+1},\dots,\ell_n}(\{u\};\{v_1,\dots,v_k\};s) = \sum_{\ell_k\neq\ell_{k+1},\dots,\ell_n} G^{(k-1)}_{\ell_k,\dots,\ell_n}(\{u\};\{v_1,\dots,v_{k-1}\};s)$$
  $$\times \underbrace{\langle\,\ell_k,\dots,\ell_n\,|\widetilde{B}(v_k;s-k+1)|\,\ell_{k+1},\dots,\ell_n\,\rangle}_{\text{easy to compute in the } F\text{-basis}},$$

⇝ recursively modifies Rosengren's sum over determinants (at each step, use of Bethe equations for $\{u\},\omega_u$)

⇝ results into additional sum over subsets of $1,\dots,n$

“Partial” vs “total” scalar products

- “partial scalar product” (for generic η):

$$S_n(\{u\};\{v\};s) = f(\{u\},\{v\},\gamma,s) \sum_{S,\tilde{S}\subset\{1,\dots,n\}} (-1)^{|S|+|\tilde{S}|} \frac{[\gamma + s - |S| + |\tilde{S}|]}{[s - |S| + |\tilde{S}|]}$$

$$\times \prod_{j\notin\tilde{S}} \left\{ \frac{a(v_j)}{d(v_j)} \prod_{t=1}^{n} [u_t - v_j + 1] \right\} \prod_{j\in\tilde{S}} \left\{ \omega_u^{-2} \prod_{t=1}^{n} [u_t - v_j - 1] \right\} \det_n \frac{[u_i - \xi_j^{S\tilde{S}} + \gamma]}{[\gamma][u_i - \xi_j^{S\tilde{S}}]}$$

$$\text{with } \xi_k^{S\tilde{S}} = \begin{cases} \xi_k - 1 & \text{if } k \in S \text{ and } k \notin \tilde{S} \\ \xi_k + 1 & \text{if } k \notin S \text{ and } k \in \tilde{S} \\ \xi_k & \text{otherwise} \end{cases} \qquad (\gamma \text{ arbitrary}).$$

"Partial" vs "total" scalar products

- "partial scalar product" (for generic η):

$$S_n(\{u\};\{v\};s) = f(\{u\},\{v\},\gamma,s) \sum_{S,\tilde{S}\subset\{1,\ldots,n\}} (-1)^{|S|+|\tilde{S}|} \frac{[\gamma+s-|S|+|\tilde{S}|]}{[s-|S|+|\tilde{S}|]}$$

$$\times \prod_{j\notin\tilde{S}} \left\{ \frac{a(v_j)}{d(v_j)} \prod_{t=1}^{n} [u_t - v_j + 1] \right\} \prod_{j\in\tilde{S}} \left\{ \omega_u^{-2} \prod_{t=1}^{n} [u_t - v_j - 1] \right\} \det_n \frac{[u_i - \xi_j^{S\tilde{S}} + \gamma]}{[\gamma][u_i - \xi_j^{S\tilde{S}}]}$$

with $\xi_k^{S\tilde{S}} = \begin{cases} \xi_k - 1 & \text{if } k\in S \text{ and } k\notin\tilde{S} \\ \xi_k + 1 & \text{if } k\notin S \text{ and } k\in\tilde{S} \\ \xi_k & \text{otherwise} \end{cases}$ (γ arbitrary).

- "total scalar product" (cyclic case $\eta = r/L$ rational):

$$\langle\{u\},\omega_u\,|\,\{v\},\omega_v\rangle = \frac{1}{L} \sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \bar{\varphi}_{\omega_u}(s)\,\varphi_{\omega_v}(s)\,S_n(\{u\};\{v\};s)$$

⇝ the s-dependence can be simplified for $\gamma = -|u| + |v|$:

$$\bar{\varphi}_{\omega_u}(s)\,\varphi_{\omega_v}(s)\,f(\{u\},\{v\},|v|-|u|,s) = \left(\frac{\omega_v}{\omega_u}\right)^s \tilde{f}(\{u\},\{v\})$$

⇝ the sum over s can be relabelled:

$$\sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \left(\frac{\omega_v}{\omega_u}\right)^s \frac{[\gamma+s-|S|+|\tilde{S}|]}{[s-|S|+|\tilde{S}|]} = \left(\frac{\omega_v}{\omega_u}\right)^{|S|-|\tilde{S}|} \sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \left(\frac{\omega_v}{\omega_u}\right)^s \frac{[\gamma+s]}{[s]}$$

“Partial” vs “total” scalar products

- “partial scalar product” (for generic η):

$$S_n(\{u\};\{v\};s) = f(\{u\},\{v\},\gamma,s) \sum_{S,\tilde{S}\subset\{1,\dots,n\}} (-1)^{|S|+|\tilde{S}|} \frac{[\gamma+s-|S|+|\tilde{S}|]}{[s-|S|+|\tilde{S}|]}$$

$$\times \prod_{j\notin\tilde{S}} \left\{ \frac{a(v_j)}{d(v_j)} \prod_{t=1}^{n}[u_t-v_j+1] \right\} \prod_{j\in\tilde{S}} \left\{ \omega_u^{-2} \prod_{t=1}^{n}[u_t-v_j-1] \right\} \det_n \frac{[u_i-\xi_j^{S\tilde{S}}+\gamma]}{[\gamma][u_i-\xi_j^{S\tilde{S}}]}$$

with $\xi_k^{S\tilde{S}} = \begin{cases} \xi_k - 1 & \text{if } k\in S \text{ and } k\notin\tilde{S} \\ \xi_k+1 & \text{if } k\notin S \text{ and } k\in\tilde{S} \\ \xi_k & \text{otherwise} \end{cases}$ (γ arbitrary).

- “total scalar product” (cyclic case $\eta = r/L$ rational):

$$\langle \{u\},\omega_u \,|\, \{v\},\omega_v\rangle = \left\{ \frac{1}{L} \sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \frac{\omega_v^s}{\omega_u^s} \frac{[\gamma+s]}{[s]} \right\} \frac{\prod_{t=1}^{n} d(u_t)\cdot \det_n\left[\Omega_\gamma(\{u\};\{v\})\right]}{\prod_{j<k}[u_j-u_k][v_k-v_j]},$$

with $\gamma = -|u|+|v|$ and

$$[\Omega_\gamma(\{u\};\{v\})]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i-v_j+\gamma]}{[u_i-v_j]} - \frac{\omega_v}{\omega_u} \frac{[u_i-v_j+\gamma+1]}{[u_i-v_j+1]} \right\} a(v_j) \prod_{t=1}^{n}[u_t-v_j+1]$$
$$+ \frac{1}{[\gamma]} \left\{ \frac{[u_i-v_j+\gamma]}{[u_i-v_j]} - \frac{\omega_u}{\omega_v} \frac{[u_i-v_j+\gamma-1]}{[u_i-v_j-1]} \right\} \omega_u^{-2} d(v_j) \prod_{t=1}^{n}[u_t-v_j-1].$$

"Square of the norm"

$$\langle\{u\},\omega_u\,|\,\{u\},\omega_u\rangle = \frac{1}{(-[0]')^n}\,\frac{\prod_{t=1}^n d(u_t)}{\prod_{j\neq k}[u_j-u_k]}\cdot \det_n\left[\frac{\partial}{\partial u_k}\mathcal{Y}_{\omega_u}(u_j|\{u\})\right]$$

where

$$\mathcal{Y}_\omega(v|\{u\}) = a(v)\prod_{t=1}^n[u_t-v+1] + \omega^{-2}d(v)\prod_{t=1}^n[u_t-v-1]$$

$$\langle\{u\},\omega_u\,|\,\{u\},\omega_u\rangle = \frac{1}{(-[0]')^n}\,\frac{\prod_{t=1}^n a(u_t)d(u_t)\,\prod_{j,k=1}^n[u_j-u_k+1]}{\prod_{j\neq k}[u_j-u_k]}\cdot \det_n\left[\Phi(\{u\})\right]$$

where

$$\left[\Phi(\{u\})\right]_{ij} = \delta_{ij}\left\{\log'\frac{a}{d}(u_i) + \sum_{t=1}^n K(u_i-u_t)\right\} - K(u_i-u_j),$$

with $\quad K(u) = \dfrac{[u-1]'}{[u-1]} - \dfrac{[u+1]'}{[u+1]}$

Remark. The expression does not depend on L

$\longrightarrow$ valid for generic η

“Partial” scalar product for $\eta = r/L$

The partial scalar product is not a scalar product of Bethe-type states (function δ_s instead of $\bar{\varphi}_{\omega_u}\,\varphi_{\omega_v}$):

$$S_n(\{u\};\{v\};s) = \sum_{\tilde{s}\in s_0+\mathbb{Z}/L\mathbb{Z}} \delta_s(\tilde{s})\, S_n(\{u\};\{v\};\tilde{s})$$

However, in the rational case $\eta = r/L$, the space of discrete, L-periodic numerical functions of s is L-dimensional with a basis given by functions of the type $\bar{\varphi}_{\omega_u}\,\varphi_{\omega}$

$\rightsquigarrow$ partial scalar product can be expressed as a sum of L determinants

This can alternatively be seen by means of an identity for theta functions:

$$\frac{[u+\gamma]\,[0]'}{[u]\,[\gamma]} = \sum_{k=0}^{L-1} e^{2\pi i\eta k u} \frac{[Lu+\gamma+k\frac{\tau}{\eta}]_{L}\,[0]'_{L}}{[Lu]_{L}\,[\gamma+k\frac{\tau}{\eta}]_{L}} \qquad \text{with } [u]_{L} = \theta_1(\eta u; L\tau)$$

applied to $u = s - |S| + |\tilde{S}|$:

$$\frac{[\gamma+s-|S|+|\tilde{S}|]}{[s-|S|+|\tilde{S}|]} = \sum_{k=0}^{L-1} e^{2\pi i\eta k(s-|S|+|\tilde{S}|)} \frac{[Ls_0+\gamma+k\frac{\tau}{\eta}]_{L}\,[0]'_{L}}{[Ls_0]_{L}\,[\gamma+k\frac{\tau}{\eta}]_{L}}$$

"Partial" scalar product for $\eta = r/L$

$$\frac{[\gamma + s - |S| + |\tilde{S}|]}{[s - |S| + |\tilde{S}|]} = \sum_{k=0}^{L-1} e^{2\pi i \eta k (s - |S| + |\tilde{S}|)} \frac{[Ls_0 + \gamma + k\frac{\tau}{\eta}]_L \, [0]'_L}{[Ls_0]_L \, [\gamma + k\frac{\tau}{\eta}]_L}$$

$$S_n(\{u\};\{v\};s) \propto \sum_{S,\tilde{S} \subset \{1,\dots,n\}} (-1)^{|S|+|\tilde{S}|} \frac{[\gamma + s - |S| + |\tilde{S}|]}{[s - |S| + |\tilde{S}|]}$$

$$\times \prod_{j \notin \tilde{S}} \left\{ \frac{a(v_j)}{d(v_j)} \prod_{t=1}^{n} [u_t - v_j + 1] \right\} \prod_{j \in \tilde{S}} \left\{ \omega_u^{-2} \prod_{t=1}^{n} [u_t - v_j - 1] \right\} \det_n \frac{[u_i - \xi_j^{S\tilde{S}} + \gamma]}{[\gamma][u_i - \xi_j^{S\tilde{S}}]}$$

can be rewritten as

$$S_n(\{u\};\{v\};s) \propto \sum_{\ell=0}^{L-1} q^{\ell s} \frac{[Ls_0 + \gamma + \ell\frac{\tau}{\eta}]_L \, [0]'_L}{[Ls_0]_L \, [\gamma + \ell\frac{\tau}{\eta}]_L} \det_n \left[\Omega_\gamma^{(\ell)}(\{u\};\{v\})\right],$$

with $q = e^{2\pi i \eta}$ and $[u]_L = \theta_1(\eta u; L\tau)$, and

$$\left[\Omega_\gamma^{(\ell)}(\{u\};\{v\})\right]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - q^{-\ell} \frac{[u_i - v_j + \gamma + 1]}{[u_i - v_j + 1]} \right\} a(v_j) \prod_{t=1}^{n} [u_t - v_j + 1]$$

$$+ \frac{(-1)^{rk}}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - q^{\ell} \frac{[u_i - v_j + \gamma - 1]}{[u_i - v_j - 1]} \right\} \omega_u^{-2} d(v_j) \prod_{t=1}^{n} [u_t - v_j - 1].$$

cf. Rosengren's formula for Z_N at $\eta = r/L$

Determinant representation for finite-size form factors

solution of the quantum inverse problem:

$$E_i^{++} = \prod_{k=1}^{i-1} \widehat{t}(\xi_k) \cdot \widehat{A}(\xi_i) \cdot \prod_{k=i}^{1} \left[\widehat{t}(\xi_k)\right]^{-1}$$

$$E_i^{--} = \prod_{k=1}^{i-1} \widehat{t}(\xi_k) \cdot \widehat{D}(\xi_i) \cdot \prod_{k=i}^{1} \left[\widehat{t}(\xi_k)\right]^{-1}$$

$\rightsquigarrow$ express form factors in terms of scalar products

These are not a priori scalar products of Bethe-type states but the s-dependent part can again be simplified !

$\rightsquigarrow$ representation in terms of determinants:

$$\langle \{u\}, \omega_u \,|\, \sigma_i^z \,|\, \{v\}, \omega_v \rangle = \left\{ \prod_{k=1}^{i-1} \frac{\tau(\xi_k, \{u\}, \omega_u)}{\tau(\xi_k, \{v\}, \omega_v)} \right\} \left\{ \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \omega_u^{-s} \omega_v^{s} \frac{[\gamma + s]}{[s]} \right\}$$

$$\times \frac{\prod_{t=1}^{n} d(u_t)}{\prod_{k<l} [u_k - u_l][v_l - v_k]} \, \det_n \left[\Omega_\gamma(\{u\};\{v\}) - 2\mathcal{P}_\gamma(\{u\};\{v\}|\xi_i) \right]$$

with $\gamma = |v| - |u|$ and $\mathcal{P}_\gamma(\{u\};\{v\}|\xi_i)$ is a rank 1 matrix

Remark. The s-dependent part can no longer be simplified if acting with several local operators $\rightsquigarrow$ more complicated formulas

Generating function for two-point function

$$\langle \mathcal{Q}^{\kappa}_{1,m} \rangle = \frac{\langle \{u\}, \omega_u \,|\, \mathcal{Q}^{\kappa}_{1,m} \,|\, \{u\}, \omega_u \rangle}{\langle \{u\}, \omega_u \,|\, \{u\}, \omega_u \rangle}, \qquad (1)$$

where $|\,\{u\}, \omega_u\,\rangle$ is a ground state of the transfer matrix, and

$$\mathcal{Q}^{\kappa}_{1,m} = \prod_{j=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \sigma_j^z \right) = \prod_{j=1}^{m} (E_j^{11} + \kappa E_j^{22})$$

$\rightsquigarrow$ it gives the probability that two sites on a same vertical line at distance m have a difference of height $\ell \leq m$ (given by the coefficient of $\kappa^{\frac{m-\ell}{2}}$ of the κ-polynomial (1))

- solution of the quantum inverse problem: $\mathcal{Q}^{\kappa}_{1,m} = \prod_{i=1}^{m} \widehat{t}_{\kappa}(\xi_i) \prod_{i=m}^{1} \left[\widehat{t}(\xi_i)\right]^{-1}$ with $\widehat{t}_{\kappa} = \widehat{A} + \kappa \widehat{D}$
- for $\eta = r/L$ and L large enough, the eigenstates $|\,\{v\}_{\kappa}, \omega_v\,\rangle$ (for $n = N/2$) of $\widehat{t}_{\kappa}(u)$ form a complete basis of the space of states $\mathrm{Fun}(\mathcal{H}[0])$ (Felder, Tarasov & Varchenko 96)
- sum over form factors:

$$\langle \mathcal{Q}^{\kappa}_{1,m} \rangle = \sum_{\{v\}_{\kappa}, \omega_v} \frac{\langle \{u\}, \omega_u \,|\, \{v\}_{\kappa}, \omega_v \rangle \, \langle \{v\}_{\kappa}, \omega_v \,|\mathcal{Q}^{\kappa}_{1,m}|\, \{u\}, \omega_u \rangle}{\langle \{u\}, \omega_u \,|\, \{u\}, \omega_u \rangle \, \langle \{v\}_{\kappa}, \omega_v \,|\, \{v\}_{\kappa}, \omega_v \rangle}$$

$$= \sum_{\{v\}_{\kappa}, \omega_v} \prod_{i=1}^{m} \frac{\tau_{\kappa}(\xi_i; \{v\}_{\kappa}, \omega_v)}{\tau(\xi_i; \{u\}, \omega_u)} \, \frac{\langle \{u\}, \omega_u \,|\, \{v\}_{\kappa}, \omega_v \rangle \, \langle \{v\}_{\kappa}, \omega_v \,|\, \{u\}, \omega_u \rangle}{\langle \{u\}, \omega_u \,|\, \{u\}, \omega_u \rangle \, \langle \{v\}_{\kappa}, \omega_v \,|\, \{v\}_{\kappa}, \omega_v \rangle}$$

Master equation for the generating function

- use determinant representation for scalar products
- replace sums by contour integrals:

$$\langle \mathcal{Q}^{\kappa}_{1,m} \rangle = \frac{1}{n!} \frac{([0]')^{2n}}{L^2} \sum_{\omega} \oint_{\Gamma(\{v\}_\kappa)} \frac{d^n z}{(2i\pi)^n} \prod_{i=1}^{m} \frac{\tau_\kappa(\xi_i; \{z\}, \omega)}{\tau(\xi_i; \{u\}, \omega_u)}$$
$$\times \left\{ \sum_{s \in \mathbf{C}^{\mathbf{L}}_{\mathbf{s_0}}} \frac{\omega^s}{\omega_u^s} \frac{[\gamma_z + s]}{[s]} \right\} \left\{ \sum_{s \in \mathbf{C}^{\mathbf{L}}_{\mathbf{s_0}}} \frac{\omega_u^s}{\omega^s} \frac{[-\gamma_z + s]}{[s]} \right\}$$
$$\times \frac{\det_n \left[\Omega_{\gamma_z}(\{u\}; \{z\} | \{u\}) \right] \cdot \det_n \left[\Omega^{(\kappa)}_{-\gamma_z}(\{z\}; \{u\} | \{z\}) \right]}{\det_n \left[\frac{\partial}{\partial u_k} \mathcal{Y}_{1;\omega_u}(u_j; \{u\}) \right] \cdot \prod_{j=1}^{n} \mathcal{Y}_{\kappa;\omega}(z_j; \{z\})} \qquad (2)$$

with $\gamma_z = |z| - |u|$.
The sum in (2) is taken over all $\omega \in \mathbb{C}$ such that $(-1)^{rn}\omega^L = 1$, and the integration contour is such that it surrounds (with index 1) all poles corresponding to solutions $\{v\}_\kappa$ with $n = N/2$ of the κ-twisted Bethe equations associated with ω.

Going further: local height probabilities

What about explicitly height-dependent quantities such as (multi-point) local height probabilities ?

$$\mathbf{P}_{\alpha_1,\ldots,\alpha_m}(s) = \Bigg(\sum_{|\,\Psi_g\,\rangle}\Bigg) \frac{\langle\,\Psi_g\,|\,\delta_{\mathbf{s}}\, E_1^{\alpha_1,\alpha_1}\ldots E_m^{\alpha_m,\alpha_m}\,|\,\Psi_g\,\rangle}{\langle\Psi_g\,|\,\Psi_g\rangle}$$

$=$ probability to find a sequence of heights

$s, s+\alpha_1, s+\alpha_1+\alpha_2, \ldots, s+\alpha_1+\alpha_2+\cdots+\alpha_m$

on m successive sites of a given vertical line of the lattice

⇝ acting with local operators on the ground state $|\,\Psi_g\,\rangle$, one obtains a new state (which is no more of Bethe type)

⇝ multiple sums over partial scalar products

⇝ additional complexity with respect to elementary building blocks of XXZ: extra sum (L terms) + deformation of the determinant

$$\mathbf{P}_{\alpha_1,\ldots,\alpha_m}(s) = \sum_{\{b_p\}} G^{\gamma}_{\alpha_1,\ldots,\alpha_m}(s;\{v_{b_p}\},\{\xi\}) \sum_{\ell=0}^{L-1} q^{\ell\mathbf{s}}\, a_\gamma^{(\ell)}(s_0)\, \frac{\det_n\left[\Phi^{(\ell)}_{\gamma;\{b_p\}}(\{v\})\right]}{\det_n[\Phi(\{v\})]}$$

To be continued...

- **Summary of the first part** (this talk)
  - ⋆ Partial scalar product as sums of determinants for generic η (can be reduced to a sum of L terms for $\eta = r/L$)
  - ⋆ Determinant representation for scalar products/ "norms" of Bethe states ($\eta = r/L$)
  - ⋆ Determinant representation for finite-size form factors ($\eta = r/L$)
  - ⋆ Master equation for the generating function of the two-point function ($\eta = r/L$)
  - ⋆ Local height probabilities as multiple sums of ratios of determinants (a priori more complicated structure than in XXZ)

- **Second part:** Damien's talk tomorrow

To be continued...

- **Synopsis of the second part** (Damien's talk tomorrow)
  ⇝ computation of several physical quantities for the cyclic SOS model at the thermodynamic limit ($\eta = r/L$, $N \to \infty$, L remains finite)
  - ⋆ description of the $2(L-r)$ degenerated ground states at the thermodynamic limit
  - ⋆ thermodynamic limit of the σ^z form factor
    ⇝ computation of spontaneous staggered polarizations (proof of a conjecture of Date et al.)
  - ⋆ one-point local height probabilities at the thermodynamic limit
    ⇝ coincides with results of Pearce and Seaton
  - ⋆ multi-point local height probabilities for CSOS model at the thermodynamic limit ⇝ multiple integral representation:

$$\mathbf{P}_{\alpha_1,\ldots,\alpha_m}(s) = \prod_{j=1}^{m}\left(\int_{\mathcal{C}_j} dz_j\right) \underbrace{G_{\alpha_1,\ldots,\alpha_m}(\mathbf{s};\{z\},\{\xi\})}_{\text{algebraic part}} \times \underbrace{\bar{\mathcal{S}}_m(\{z\};\{\xi\})}_{\text{determinant contribution}} \cdot \underbrace{\bar{\mathbf{P}}(s,|z|-|\xi|)}_{\text{deformed 1-point LHP}},$$

Do not miss Damien's talk tomorrow!